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Generalized Christoffel–Darboux formula for classical skew-orthogonal polynomials

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Abstract

We show that skew-orthogonal functions, defined with respect to Jacobi weight $w_{a,b}(x) = (1-x)^a(1+x)^b$, $a, b > -1$, including the limiting cases of Laguerre ($w_a(x) = x^a e^{-x}$, $a > -1$) and Gaussian weight ($w(x) = e^{-x^2}$), satisfy three-term recursion relation in the quaternion space. From this, we derive generalized Christoffel–Darboux (GCD) formulae for kernel functions arising in the study of the corresponding orthogonal and symplectic ensembles of random $2N \times 2N$ matrices. Using the GCD formulae we calculate the level densities and prove that in the bulk of the spectrum, under appropriate scaling, the eigenvalue correlations are universal. We also provide evidence to show that there exists a mapping between skew-orthogonal functions arising in the study of orthogonal and symplectic ensembles of random matrices.

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1. Introduction

1.1. Random matrices

Universality of eigenvalue correlations for different random matrix ensembles and its application in real physical systems has attracted both mathematicians and physicists in the last few decades [1–5, 9–17]. From the mathematical point of view, the study of this universal behavior of eigenvalue correlations for various ensembles of large random matrices require (i) evaluation of certain kernel functions involving orthogonal (unitary ensemble), skew-orthogonal (orthogonal and symplectic ensemble) and bi-orthogonal (unitary two-matrix ensemble) polynomials, (ii) asymptotic analysis of these polynomials. The rich literature available on orthogonal [9–21] and bi-orthogonal polynomials [22–26] corresponding to different weights have contributed a lot in our understanding of unitary ensembles.

For orthogonal and symplectic ensembles, the problem of evaluating the kernel functions is removed by the Tracy–Widom formalism, where the kernel functions are expressed in terms of that of the unitary ensemble, involving orthogonal polynomials [38–42]. Results for orthogonal polynomials (OP) are then used.

The second and perhaps a more enriching method is to evaluate the kernel functions directly in terms of skew-orthogonal polynomials/skew-orthogonal functions (SOP) and use asymptotic properties of these functions [27]. For this, we need to develop the theory of SOP [3–8, 28–32] so that we can have further insight into orthogonal and symplectic ensembles of random matrices.

In this paper, we study statistical properties of orthogonal and symplectic ensembles of random matrices with classical weight using SOP. To do so, we evaluate the kernel functions (which we have termed generalized Christoffel–Darboux sum or GCD) for the corresponding orthogonal and symplectic ensembles of random $2N \times 2N$ matrices. As $N \rightarrow \infty$, study of these kernel functions require a knowledge of the asymptotic behavior of the corresponding SOP. These can be derived by solving the Riemann–Hilbert problem [5, 45] for SOP. The other (and perhaps easier) option is to obtain a finite term recursion relation between SOP and OP and use known properties of the latter (also see [3, 4]) to obtain asymptotic results for SOP. We use the second option.

OP satisfy a three-term recursion relation, irrespective of the weight function $w(x)$ w.r.t. which they are defined. However recursion relations satisfied by SOP depend on the corresponding weight. This is because these functions, defined in the range $[x_1, x_2]$, show skew-orthonormality w.r.t. their derivatives (1.16). This brings into picture terms like $w'(x)/w(x)$ which gives rise to certain local behavior of the recursion relations. For example, for SOP defined in the finite range $[x_1, x_2]$, $w'(x)/w(x)$ may have poles at the end points, which have to be dealt with in the recursion relations. For SOP defined in the infinite range $[-\infty, \infty]$, this can increase the number of terms in the recursion relations.

An obvious consequence of the local behavior of recursion relations is that the GCD formulae also depend on $w(x)$. In this paper, we calculate the GCD sum corresponding to Jacobi weight, including the limiting cases of associated Laguerre and Gaussian weight. GCD formula for weight functions with polynomial potential has been derived in [5] and is mentioned in this paper for completeness. In [3, 4, 28], the authors have found compact expressions for SOP in terms of OP. In this paper, we use them and make further developments of the asymptotic analysis of these SOP. Finally, using these asymptotic results in the GCD formulae, we study the statistical properties of the corresponding random matrix ensembles in the bulk of the spectrum.

We also observe certain duality property between the two families of SOP arising in the study of orthogonal and symplectic ensembles. (In fact, this justifies further the use of SOP to ordinary OP in studying these ensembles [9, 10, 33–41].) However, this can at best be termed in physics literature as ‘experimental observations’. We do not have a clear theoretical understanding of this duality. The answer perhaps lies in the existence of certain ortho-symplectic group which shares such duality property. But the author’s knowledge in this field is severely limited.

We consider ensembles of $2N$ -dimensional matrices H with probability distribution

$$P_{\beta,N}(H) dH = \frac{1}{\mathcal{Z}_{\beta,N}} \exp[-2 \text{Tr } u(H)] dH, \quad (1.1)$$

where the matrix function $u(H)$ is defined by the power expansion of the function $u(z)$. The parameter $\beta = 1, 2$ and 4 corresponds to ensembles invariant under orthogonal, unitary and

symplectic transformations. In this paper, we study $\beta = 1$ and 4 cases. The partition function is given by

$$Z_{\beta,N} := \int_{H \in M_{2N}^{(\beta)}} \exp[-2 \operatorname{Tr} u(H)] dH = (2N)! \prod_{j=0}^{2N-1} g_j^{(\beta)}, \quad (1.2)$$

where $M_{2N}^{(\beta)}$ is a set of all $2N \times 2N$ real symmetric ($\beta = 1$) and quaternion real self-dual ($\beta = 4$) matrices. dH is the standard Haar measure. $g_j^{(\beta)}$ are normalization constants for SOP [27] corresponding to $\beta = 1$ and 4.

1.2. Skew-orthogonal polynomials: relevance in orthogonal and symplectic ensembles

From the invariance of these ensembles under orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) transformation, joint probability of eigenvalues $(x_1, x_2, \dots, x_{2N})$ is given by [28]

$$P_{\beta,N}(x_1, x_2, \dots, x_{2N}) = \frac{1}{Z_{\beta,N}} |\Delta_{2N}(x_1, x_2, \dots, x_{2N})|^\beta \prod_{j=1}^{2N} w(x_j), \quad (1.3)$$

where $\Delta_{2N}(x_1, x_2, \dots, x_{2N}) = \prod_{j < k} (x_j - x_k)$ is the Vandermonde determinant. The weight function $w(x) = \exp[-u(x)]$ is a nonzero and non-negative function on the interval $[a, b]$ and have finite moments.

The n -point correlation of eigenvalues is given by

$$R_n^{(\beta)}(x_1, \dots, x_n) = \frac{2N!}{(2N - n)!} \int dx_{n+1} \cdots \int dx_{2N} P_{\beta,N}(x_1, x_2, \dots, x_{2N}), \quad n = 1, 2, \dots \quad (1.4)$$

To evaluate such integrals, the joint probability distribution $P_{\beta,N}(x_1, x_2, \dots, x_{2N})$ is written in terms of quaternion determinants (i.e. determinant of a matrix, each of whose element is a 2×2 quaternion) satisfying certain properties [29]. Finally, using Dyson–Mehta theorem (page 152 of [29]), one can calculate (1.4). For example, the two-point function $R_2^{(\beta)}(x, y)$ and the level density $R_1^{(\beta)}(x)$ is given by

$$R_2^{(\beta)}(x, y) = \begin{pmatrix} S_{2N}^{(\beta)}(x, y) & D_{2N}^{(\beta)}(x, y) \\ I_{2N}^{(\beta)}(x, y) - \delta_{1,\beta} \epsilon(y - x) & S_{2N}^{(\beta)}(y, x) \end{pmatrix}; \quad (1.5)$$

$$R_1^{(\beta)}(x) := \rho^{(\beta)}(x) = S_{2N}^{(\beta)}(x, x), \quad \epsilon(r) = \frac{|r|}{2r}.$$

Here δ is the kronecker delta. In terms of SOP $\phi_n^{(\beta)}(x)$ and $\psi_n^{(\beta)}(x)$, to be defined in (1.10) and (1.15) respectively, they are expressed as

$$S_{2N}^{(\beta)}(x, y) := \sum_{j,k=0}^{2N-1} Z_{j,k} \phi_j^{(\beta)}(x) \psi_k^{(\beta)}(y) = \widehat{\Psi}^{(\beta)}(y) \prod_{2N} \Phi^{(\beta)}(x) \\ = -\widehat{\Phi}^{(\beta)}(x) \prod_{2N} \Psi^{(\beta)}(y), \quad (1.6)$$

$$D_{2N}^{(\beta)}(x, y) := - \sum_{j,k=0}^{2N-1} Z_{j,k} \phi_j^{(\beta)}(x) \phi_k^{(\beta)}(y) = \widehat{\Phi}^{(\beta)}(x) \prod_{2N} \Phi^{(\beta)}(y), \quad (1.7)$$

$$I_{2N}^{(\beta)}(x, y) := \sum_{j,k=0}^{2N-1} Z_{j,k} \psi_j^{(\beta)}(x) \psi_k^{(\beta)}(y) = -\widehat{\Psi}^{(\beta)}(x) \prod_{2N} \Psi^{(\beta)}(y), \quad (1.8)$$

$$S_{2N}^{(\beta)}(y, x) = S_{2N}^{\dagger(\beta)}(x, y) = \widehat{\Phi}^{(\beta)}(x) \prod_{2N} \Psi^{(\beta)}(y), \tag{1.9}$$

where

$$\phi_n^{(\beta)}(x) = \frac{1}{\sqrt{g_n^{(\beta)}}} \pi_n^{(\beta)}(x) w(x), \quad \pi_n^{(\beta)}(x) = \sum_{k=0}^n c_k^{(n,\beta)} x^k, \quad \beta = 1, 4, \tag{1.10}$$

are normalized SOP of order n .

Here, $\prod_{2N} = \text{diag}(\underbrace{1, \dots, 1}_{2N}, 0, \dots, 0)$ is a diagonal matrix and $\Phi^{(\beta)}(x)$ and $\Psi^{(\beta)}(x)$ are semi-infinite vectors

$$\Phi^{(\beta)}(x) = (\Phi_0^{(\beta)t}(x) \dots \Phi_n^{(\beta)t}(x) \dots)^t, \quad \widehat{\Phi}^{(\beta)}(x) = -\Phi^{(\beta)t}(x)Z, \tag{1.11}$$

$$\Psi^{(\beta)}(x) = (\Psi_0^{(\beta)t}(x) \dots \Psi_n^{(\beta)t}(x) \dots)^t, \quad \widehat{\Psi}^{(\beta)}(x) = -\Psi^{(\beta)t}(x)Z, \tag{1.12}$$

with each entry a 2×1 matrix

$$\Phi_n^{(\beta)}(x) = \begin{pmatrix} \phi_{2n}^{(\beta)}(x) \\ \phi_{2n+1}^{(\beta)}(x) \end{pmatrix}, \quad \widehat{\Phi}_n^{(\beta)}(x) = \begin{pmatrix} \phi_{2n+1}^{(\beta)}(x) \\ -\phi_{2n}^{(\beta)}(x) \end{pmatrix}^t \tag{1.13}$$

(similar for $\Psi_n^{(\beta)}(x)$ and $\widehat{\Psi}_n^{(\beta)}(x)$). The anti-symmetric block-diagonal matrix Z is given by

$$Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{+} \dots \dot{+} \tag{1.14}$$

such that $Z = -Z^t$ and $Z^2 = -1$.

For

$$\Psi_n^{(4)}(x) = \Phi_n^{(4)}(x), \quad \Psi_n^{(1)}(x) = \int_{\mathbb{R}} \Phi_n^{(1)}(y) \epsilon(x - y) dy, \quad n \in \mathbb{N}, \tag{1.15}$$

these polynomials satisfy skew-orthonormal relations w.r.t. the weight function $w^2(x)$,¹

$$(\Phi_n^{(\beta)}, \widehat{\Psi}_m^{(\beta)}) \equiv \int_{\mathbb{R}} \Phi_n^{(\beta)}(x) \widehat{\Psi}_m^{(\beta)}(x) dx = \delta_{nm} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad n, m \in \mathbb{N}. \tag{1.16}$$

Finally, from (1.6)–(1.8) we get

$$D_{2N}^{(1)}(x, y) = -\frac{\partial S_{2N}^{(1)}(x, y)}{\partial y}, \quad S_{2N}^{(1)}(x, y) = \frac{\partial I_{2N}^{(1)}(x, y)}{\partial y}, \tag{1.17}$$

$$I_{2N}^{(4)}(x, y) = \frac{\partial S_{2N}^{(4)}(x, y)}{\partial x}, \quad S_{2N}^{(4)}(x, y) = -\frac{\partial D_{2N}^{(4)}(x, y)}{\partial x}. \tag{1.18}$$

Thus a knowledge of the kernel function $S_{2N}^{(\beta)}(x, y)$ is enough to calculate the correlation function. In this paper, we will study the finite N and large N behavior of $S_{2N}^{(\beta)}(x, y)$.

The outline of the paper is as follows. In section 2, we calculate GCD formulae for the kernel function $S_{2N}^{(\beta)}(x, y)$ (1.6), $\beta = 1$ and 4, corresponding to different weight. In section 3, we discuss the idea of duality that exists between the two families of SOP arising in the study

¹ To observe the dual property among the two families of polynomials $\pi_n^{(\beta)}(x)$, $\beta = 1, 4$, we skew-orthonormalize them w.r.t. $w^2(x)$ (as in 1.16) to set both the families of SOP on an equal footing. However to study the statistical properties of symplectic ensembles only, this is not needed.

of orthogonal and symplectic ensembles of random matrices. In section 4, we give a brief summary of some of the relevant properties of classical OP which will be useful in our study of the corresponding SOP. In section 5, we use results of section 2 to (i) obtain the level densities (1.5) for Jacobi and associated Laguerre orthogonal ensembles, (ii) prove that in the bulk of the spectrum, the kernel functions $S_{2N}^{(\beta)}(x, y)/S_{2N}^{(\beta)}(x, x)$ and hence the unfolded correlation functions for the above ensembles are stationary and universal. In section 6, we repeat the same calculations for Jacobi symplectic and associated Laguerre symplectic ensembles².

2. The generalized Christoffel Darboux sum

2.1. Recursion relations

For polynomials with weight function

$$w(x) = (x_2 - x)^a (x - x_1)^b, \quad x_1, x_2 \in \mathbb{R}, \quad (2.1)$$

skew-orthogonal in the finite interval $[x_1, x_2]$, and having finite moments, evaluation of $\psi_n^{(4)}(x)$ and $\phi_n^{(1)}(x)$ will involve terms like $w'(x)/w(x)$ which have poles at x_1 and x_2 . Hence to obtain recursion relations we expand $[(x - x_1)(x_2 - x)\Phi^{(\beta)}(x)]'$ and $[x(x - x_1)(x_2 - x)\Phi^{(\beta)}(x)]'$ in terms of SOP $\Phi^{(\beta)}(x)$ (1.11) and introduce semi-infinite matrices $P^{(\beta)}$ and $R^{(\beta)}$ such that for $\beta = 4$,

$$(x - x_1)(x_2 - x)(\Phi^{(4)}(x))' \equiv f(x)(\Phi^{(4)}(x))' = P^{(4)}\Phi^{(4)}(x), \quad (2.2)$$

$$x(x - x_1)(x_2 - x)(\Phi^{(4)}(x))' \equiv xf(x)(\Phi^{(4)}(x))' = R^{(4)}\Phi^{(4)}(x). \quad (2.3)$$

For $\beta = 1$, we get

$$(x - x_1)(x_2 - x)\Phi^{(1)}(x) \equiv f(x)\Phi^{(1)}(x) = P^{(1)}\Psi^{(1)}(x), \quad (2.4)$$

$$x(x - x_1)(x_2 - x)\Phi^{(1)}(x) \equiv xf(x)\Phi^{(1)}(x) = R^{(1)}\Psi^{(1)}(x). \quad (2.5)$$

Equations (2.4) and (2.5) are obtained by multiplying the above expansion by $\epsilon(y - x)$ and integrating by parts.

In this context, the Jacobi weight function is defined in the interval $[-1, 1]$ by

$$w_{a,b}(x) = (1 - x)^a (1 + x)^b, \quad a > -1, \quad b > -1, \quad (2.6)$$

where restrictions on a and b ensure that they have finite moments.

Associated Laguerre weight function is defined in the interval $[0, \infty]$ by

$$w_a(x) = x^a e^{-x}, \quad a > -1, \quad (2.7)$$

where restriction on a ensures that they have finite moments.

Gaussian weight function is defined in the interval $[-\infty, \infty]$ by

$$w(x) = e^{-x^2}. \quad (2.8)$$

From here on, we will concentrate on these classical weight functions and show that the corresponding SOP satisfy three-term recursion relations in the 2×2 quaternion space.

² For random matrix ensembles with polynomial potential, the special case of Gaussian ensemble ($d = 1$), which is also one of the limiting case of Jacobi ensemble, has been worked out explicitly in [5] using GCD formula, and hence not repeated in this paper.

2.2. Recursion relations for SOP with classical weight

For classical weight, we expand $[(f(x)\Phi^{(\beta)}(x))']$ and $[x(f(x)\Phi^{(\beta)}(x))']$ in terms of $\Phi^{(\beta)}(x)$. They satisfy the following recursion relations:

$$f(x)\Psi^{(4)}(x) = P^{(4)}\Phi^{(4)}(x), \quad xf(x)\Psi^{(4)}(x) = R^{(4)}\Phi^{(4)}(x), \quad (2.9)$$

$$f(x)\Phi^{(1)}(x) = P^{(1)}\Psi^{(1)}(x), \quad xf(x)\Phi^{(1)}(x) = R^{(1)}\Psi^{(1)}(x), \quad (2.10)$$

where $P^{(\beta)}$ and $R^{(\beta)}$ are semi-infinite tridiagonal quaternion matrices. The semi-infinite vectors $\Phi^{(\beta)}(x)$ and $\Psi^{(\beta)}(x)$ are given in (1.11) and (1.12) respectively. Now, $w'(x)/w(x)$ and hence $\Psi^{(4)}(x)$ and $\Phi^{(1)}(x)$ has singularity at $x \pm 1$ for Jacobi, and at $x = 0$ for associated Laguerre. To remove them, we have

$$f(x) = (1 - x^2), \quad \text{Jacobi} \quad (2.11)$$

$$= x, \quad \text{Associated Laguerre} \quad (2.12)$$

$$= 1, \quad \text{Gaussian (also true for any polynomial weight)}. \quad (2.13)$$

In other words, SOP satisfy three-term recursion relation in the quaternion space and is given by

$$f(x)\Psi_n^{(4)}(x) = \mathbf{P}_{n,n+1}^{(4)}\Phi_{n+1}^{(4)}(x) + \mathbf{P}_{n,n}^{(4)}\Phi_n^{(4)}(x) + \mathbf{P}_{n,n-1}^{(4)}\Phi_{n-1}^{(4)}(x), \quad (2.14)$$

$$xf(x)\Psi_n^{(4)}(x) = \mathbf{R}_{n,n+1}^{(4)}\Phi_{n+1}^{(4)}(x) + \mathbf{R}_{n,n}^{(4)}\Phi_n^{(4)}(x) + \mathbf{R}_{n,n-1}^{(4)}\Phi_{n-1}^{(4)}(x). \quad (2.15)$$

$$f(x)\Phi_n^{(1)}(x) = \mathbf{P}_{n,n+1}^{(1)}\Psi_{n+1}^{(1)}(x) + \mathbf{P}_{n,n}^{(1)}\Psi_n^{(1)}(x) + \mathbf{P}_{n,n-1}^{(1)}\Psi_{n-1}^{(1)}(x), \quad (2.16)$$

$$xf(x)\Phi_n^{(1)}(x) = \mathbf{R}_{n,n+1}^{(1)}\Psi_{n+1}^{(1)}(x) + \mathbf{R}_{n,n}^{(1)}\Psi_n^{(1)}(x) + \mathbf{R}_{n,n-1}^{(1)}\Psi_{n-1}^{(1)}(x), \quad (2.17)$$

where $\Phi_n^{(\beta)}(x)$ and $\Psi_n^{(\beta)}(x)$ are given in (1.13) and $\mathbf{P}_{j,k}^{(\beta)}$ and $\mathbf{R}_{j,k}^{(\beta)}$ are 2×2 quaternions. Equations (2.14)–(2.17) can be proved directly using the skew-orthogonal relation (1.16). We leave it as an exercise. In this paper, we will give an alternative proof by showing that the semi-infinite matrices $P^{(\beta)}$ and $R^{(\beta)}$ are tridiagonal (in the quaternion sense) and anti-self-dual.

In terms of the elements of the quaternion matrices, (2.14) and (2.15) can be written as

$$f(x) \begin{pmatrix} \psi_{2n}^{(4)}(x) \\ \psi_{2n+1}^{(4)}(x) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ P_{2n+1,2n+2}^{(4)} & 0 \end{pmatrix} \begin{pmatrix} \phi_{2n+2}^{(4)}(x) \\ \phi_{2n+3}^{(4)}(x) \end{pmatrix} + \begin{pmatrix} P_{2n,2n}^{(4)} & P_{2n,2n+1}^{(4)} \\ P_{2n+1,2n}^{(4)} & P_{2n+1,2n+1}^{(4)} \end{pmatrix} \begin{pmatrix} \phi_{2n}^{(4)}(x) \\ \phi_{2n+1}^{(4)}(x) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ P_{2n+1,2n-2}^{(4)} & 0 \end{pmatrix} \begin{pmatrix} \phi_{2n-2}^{(4)}(x) \\ \phi_{2n-1}^{(4)}(x) \end{pmatrix} \quad (2.18)$$

and

$$xf(x) \begin{pmatrix} \psi_{2n}^{(4)}(x) \\ \psi_{2n+1}^{(4)}(x) \end{pmatrix} = \begin{pmatrix} R_{2n,2n+2}^{(4)} & 0 \\ R_{2n+1,2n+2}^{(4)} & R_{2n+1,2n+3}^{(4)} \end{pmatrix} \begin{pmatrix} \phi_{2n+2}^{(4)}(x) \\ \phi_{2n+3}^{(4)}(x) \end{pmatrix} + \begin{pmatrix} R_{2n,2n}^{(4)} & R_{2n,2n+1}^{(4)} \\ R_{2n+1,2n}^{(4)} & R_{2n+1,2n+1}^{(4)} \end{pmatrix} \begin{pmatrix} \phi_{2n}^{(4)}(x) \\ \phi_{2n+1}^{(4)}(x) \end{pmatrix} + \begin{pmatrix} R_{2n,2n-2}^{(4)} & 0 \\ R_{2n+1,2n-2}^{(4)} & R_{2n+1,2n-1}^{(4)} \end{pmatrix} \begin{pmatrix} \phi_{2n-2}^{(4)}(x) \\ \phi_{2n-1}^{(4)}(x) \end{pmatrix}. \quad (2.19)$$

For $\beta = 1$, we get similar relations, where $\Phi^{(4)}$ and $\Psi^{(4)}$ are replaced by $\Psi^{(1)}$ and $\Phi^{(1)}$ respectively.

For the polynomial weight, the semi-infinite matrices $P^{(\beta)}$ and $R^{(\beta)}$ have d quaternion bands above and below the diagonal [5]. Thus the Gaussian ($d = 1$) SOP, like the Jacobi and associated Laguerre functions, satisfy three-term recursion in the quaternion space.

Note. Here, we would like to mention that unlike OP, the Jacobi matrix $Q^{(\beta)}$ coming from the relation $x\Phi^{(\beta)}(x) = Q^{(\beta)}\Phi^{(\beta)}(x)$, $\beta = 1, 4$, holds little importance as they do not have finite bands below the diagonal.

2.3. Proof

To prove that the SOP corresponding to classical weight satisfy three-term recursion in the quaternion space, we will prove that the matrices $P^{(4)}$ and $R^{(4)}$ for Jacobi weight are anti-self-dual. We use the scalar products

$$\sum_j P_{n,j}^{(4)} Z_{j,n} = ((1-x^2)\psi_n^{(4)}(x), \psi_m^{(4)}(x)) = \sum_j P_{m,j}^{(4)} Z_{j,n}, \tag{2.20}$$

$$\sum_j R_{n,j}^{(4)} Z_{j,n} = (x(1-x^2)\psi_n^{(4)}(x), \psi_m^{(4)}(x)) = \sum_j R_{m,j}^{(4)} Z_{j,n}. \tag{2.21}$$

Similarly, using $(x\psi_n^{(4)}(x), \psi_m^{(4)}(x))$ and $(x^2\psi_n^{(4)}(x), \psi_m^{(4)}(x))$ for associated Laguerre weight and $(\psi_n^{(4)}(x), \psi_m^{(4)}(x))$ and $(x\psi_n^{(4)}(x), \psi_m^{(4)}(x))$ for polynomial weight for $\beta = 4$ and replacing $\psi^{(4)}(x)$ by $\phi^{(1)}(x)$ for $\beta = 1$, we get

$$P^{(\beta)} = -P^{(\beta)D}, \quad R^{(\beta)} = -R^{(\beta)D}, \tag{2.22}$$

where dual of a matrix is defined as

$$A^D := -ZA^tZ. \tag{2.23}$$

It is straightforward to see that $P^{(\beta)}$ and $R^{(\beta)}$ have finite bands (one in the case of SOP defined w.r.t. the classical weight functions) above the diagonal. Equation (2.22) ensures that they also have the same number of bands (where each entry is a 2×2 quaternion [29]) below the diagonal. This completes the proof.

2.4. Generalized Christoffel Darboux sum

In this subsection, we generalize the results given in [5] to include GCD sum for both classical weights as well as weight functions with polynomial potential.

With $f(y)$ given in (2.11), we use (1.6) and (2.9) to get

$$\begin{aligned} f(y)S_{2N}^{(4)}(x, y) - f(x)S_{2N}^{(4)}(y, x) &= f(y) \left[\Phi^{(4)t}(x) \prod_{2N} Z \prod_{2N} \Psi^{(4)}(y) \right] \\ &\quad + f(x) \left[\Psi^{(4)t}(x) \prod_{2N} Z \prod_{2N} \Phi^{(4)}(y) \right] \\ &= \left[-\Phi^{(4)t}(x) Z Z \prod_{2N} Z \prod_{2N} P^{(4)} \Phi^{(4)}(y) \right] \\ &\quad + \left[\Phi^{(4)t}(x) Z Z P^{(4)t} Z Z \prod_{2N} Z \prod_{2N} \Phi^{(4)}(y) \right] \end{aligned}$$

$$\begin{aligned}
 &= \left[-\widehat{\Phi}^{(4)}(x) \prod_{2N} P^{(4)} \Phi^{(4)}(y) \right] + \left[\widehat{\Phi}^{(4)}(x) P^{(4)} \prod_{2N} \Phi^{(4)}(y) \right] \\
 &= \widehat{\Phi}^{(4)}(x) \left[P^{(4)}, \prod_{2N} \right] \Phi^{(4)}(y).
 \end{aligned} \tag{2.24}$$

Similarly,

$$\begin{aligned}
 yf(y)S_{2N}^{(4)}(x, y) - xf(x)S_{2N}^{(4)}(y, x) &= yf(y) \left[\Phi^{(4)t}(x) \prod_{2N} Z \prod_{2N} \Psi^{(4)}(y) \right] \\
 &\quad + xf(x) \left[\Psi^{(4)t}(x) \prod_{2N} Z \prod_{2N} \Phi^{(4)}(y) \right] \\
 &= \left[-\Phi^{(4)t}(x) ZZ \prod_{2N} Z \prod_{2N} R^{(4)} \Phi^{(4)}(y) \right] \\
 &\quad + \left[\Phi^{(4)t}(x) ZZ R^{(4)t} ZZ \prod_{2N} Z \prod_{2N} \Phi^{(4)}(y) \right] \\
 &= \left[-\widehat{\Phi}^{(4)}(x) \prod_{2N} R^{(4)} \Phi^{(4)}(y) \right] + \left[\widehat{\Phi}^{(4)}(x) R^{(4)} \prod_{2N} \Phi^{(4)}(y) \right] \\
 &= \widehat{\Phi}^{(4)}(x) \left[R^{(4)}, \prod_{2N} \right] \Phi^{(4)}(y).
 \end{aligned} \tag{2.25}$$

Combining the two, GCD formula for symplectic ensembles of random matrices with classical weight is given by

$$S_{2N}^{(4)}(x, y) = \frac{\widehat{\Phi}^{(4)}(x) [\overline{R}^{(4)}(x), \prod_{2N}] \Phi^{(4)}(y)}{f(y)(y-x)}, \quad N \geq 1. \tag{2.26}$$

For the corresponding orthogonal ensembles ($\beta = 1$), GCD formula is derived using similar technique. From (1.6) and (2.10), we get

$$\begin{aligned}
 f(x)S_{2N}^{(1)}(x, y) - f(y)S_{2N}^{(1)}(y, x) &= f(x) \left[\Phi^{(1)t}(x) \prod_{2N} Z \prod_{2N} \Psi^{(1)}(y) \right] \\
 &\quad + f(y) \left[\Psi^{(1)t}(x) \prod_{2N} Z \prod_{2N} \Phi^{(1)}(y) \right] \\
 &= \widehat{\Psi}^{(1)}(x) \left[P^{(1)}, \prod_{2N} \right] \Psi^{(1)}(y),
 \end{aligned} \tag{2.27}$$

and

$$\begin{aligned}
 xf(x)S_{2N}^{(1)}(x, y) - yf(y)S_{2N}^{(1)}(y, x) &= xf(x) \left[\Phi^{(1)t}(x) \prod_{2N} Z \prod_{2N} \Psi^{(1)}(y) \right] \\
 &\quad + yf(y) \left[\Psi^{(1)t}(x) \prod_{2N} Z \prod_{2N} \Phi^{(1)}(y) \right], \\
 &= \widehat{\Psi}^{(1)}(x) \left[R^{(1)}, \prod_{2N} \right] \Psi^{(1)}(y).
 \end{aligned} \tag{2.28}$$

Combining the two, the GCD formula for classical orthogonal ensembles is given by

$$S_{2N}^{(1)}(x, y) = \frac{\widehat{\Psi}^{(1)}(x) [\overline{R}^{(1)}(y), \prod_{2N}] \Psi^{(1)}(y)}{f(x)(x - y)}, \quad N \geq 1. \quad (2.29)$$

Here

$$\overline{R}^{(\beta)}(x) = R^{(\beta)} - xP^{(\beta)}, \quad \beta = 1, 4, \quad (2.30)$$

is different for different weights.

For example, GCD matrix for the Jacobi symplectic ensemble (including the associated Laguerre and Gaussian symplectic ensemble) has the following structure:

$$\begin{aligned} \widehat{\Phi}^{(4)}(x) \left[\overline{R}^{(4)}(x), \prod_{2N} \right] \Phi^{(4)}(y) &= \begin{pmatrix} \phi_1^{(4)}(x) \\ -\phi_0^{(4)}(x) \\ \vdots \end{pmatrix}^t \\ &\times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \vdots & \vdots & 0 \\ 0 & 0 & 0 & -R_{2N-2,2N}^{(4)} & 0 & 0 \\ 0 & 0 & 0 & -\overline{R}_{2N-1,2N}^{(4)}(x) & -R_{2N-1,2N+1}^{(4)} & 0 \\ 0 & -R_{2N-1,2N+1}^{(4)} & 0 & 0 & 0 & 0 \\ 0 & \overline{R}_{2N-1,2N}^{(4)}(x) & -R_{2N-2,2N}^{(4)} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \vdots & \vdots & 0 \end{pmatrix} \\ &\times \begin{pmatrix} \phi_0^{(4)}(y) \\ \phi_1^{(4)}(y) \\ \vdots \end{pmatrix} \\ &= R_{2N-2,2N}^{(4)} [\phi_{2N}^{(4)}(x)\phi_{2N-1}^{(4)}(y) - \phi_{2N}^{(4)}(y)\phi_{2N-1}^{(4)}(x)] \\ &\quad + R_{2N-1,2N+1}^{(4)} [\phi_{2N-2}^{(4)}(x)\phi_{2N+1}^{(4)}(y) - \phi_{2N-2}^{(4)}(y)\phi_{2N+1}^{(4)}(x)] \\ &\quad + (R_{2N-1,2N}^{(4)} - xP_{2N-1,2N}^{(4)}) [\phi_{2N-2}^{(4)}(x)\phi_{2N}^{(4)}(y) - \phi_{2N-2}^{(4)}(y)\phi_{2N}^{(4)}(x)]. \quad (2.31) \end{aligned}$$

Similarly, for classical orthogonal ensemble, the GCD matrix has the following structure:

$$\begin{aligned} \widehat{\Psi}^{(1)}(x) \left[\overline{R}^{(1)}(y), \prod_{2N} \right] \Psi^{(1)}(y) &= R_{2N-2,2N}^{(1)} [\psi_{2N}^{(1)}(x)\psi_{2N-1}^{(1)}(y) - \psi_{2N}^{(1)}(y)\psi_{2N-1}^{(1)}(x)] \\ &\quad + R_{2N-1,2N+1}^{(1)} [\psi_{2N-2}^{(1)}(x)\psi_{2N+1}^{(1)}(y) - \psi_{2N-2}^{(1)}(y)\psi_{2N+1}^{(1)}(x)] \\ &\quad + (R_{2N-1,2N}^{(1)} - yP_{2N-1,2N}^{(1)}) [\psi_{2N-2}^{(1)}(x)\psi_{2N}^{(1)}(y) - \psi_{2N-2}^{(1)}(y)\psi_{2N}^{(1)}(x)]. \quad (2.32) \end{aligned}$$

3. Duality

Duality between the two families of SOP arising in the study of orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) ensembles of random matrices was predicted in [5, 6]. In this section, we show the existence of such duality between the two families of SOP corresponding to classical weight, i.e. $\Phi^{(4)}(x) \mapsto \Psi^{(1)}(x)$ and $\Psi^{(4)}(x) \mapsto \Phi^{(1)}(x)$. For this, we derive

recursion relations between the two families of SOP with their corresponding OP. Apart from demonstrating duality, this technique simplifies the derivation of asymptotic results of the SOP.

We expand functions $\Phi_m^{(4)}(x)$ and $\Psi_m^{(1)}(x)$, $m \geq 1$, skew-orthogonal in the range $[x_1, x_2]$, in a suitable basis of OP such that:

- (i) their derivatives are continuous in the range $[x_1, x_2]$ and vanish at the end-points.
- (ii) $\phi_m^{(4)}(x)$ and $(\psi_m^{(1)}(x))' \equiv \phi_m^{(1)}(x)$ can be written as $w(x)\pi_m^{(\beta)}(x)$.
- (iii) $\Phi_m^{(4)}(x)$ and $\Psi_m^{(1)}(x)$ are skew-orthonormal in the range $[x_1, x_2]$ w.r.t. their derivatives.

We expand Jacobi SOP $\phi_m^{(4)}(x)$ and $\psi_m^{(1)}(x)$ in terms of Jacobi OP $P_j^{2a+1, 2b+1}(x)$, orthogonal w.r.t. the weight function $w_{2a+1, 2b+1}(x)$ (see equation (2.6)). For associated Laguerre SOP, we expand in terms of $L_j^{2a+1}(x)$, orthogonal w.r.t. the weight function $w_{2a+1}(x)$ (see equation (2.7)) while for Gaussian SOP, the basis chosen is $H_j(x)$ orthogonal w.r.t. the weight e^{-x^2} . The choice of such basis ensures that conditions (i)–(iii) are satisfied.

3.1. Jacobi SOP

Jacobi SOP corresponding to orthogonal and symplectic ensembles are given below. We see that there exists a relation between $\psi_m^{(1)}(x)$ and $\phi_m^{(4)}(x)$ and their derivatives. We will use the following identities:

$$(1 - x^2)w_{a,b}(x) = w_{a+1,b+1}(x), \quad w_{a,b}(x)w_{a+1,b+1}(x) = w_{2a+1, 2b+1}(x). \tag{3.1}$$

For $\beta = 1$, with $\Phi_m^{(1)}(x)$ and $\Psi_m^{(1)}(x)$ satisfying conditions (i)–(iii), we have for $m \geq 1$,

$$(g_{2m}^{(1)})^{1/2} \psi_{2m+1}^{(1)}(x) = w_{a+1,b+1}(x)P_{2m}^{2a+1, 2b+1}(x), \tag{3.2}$$

$$(g_{2m}^{(1)})^{1/2} \phi_{2m+1}^{(1)}(x) = w_{a,b}(x)[A_{2m+1}P_{2m+1}^{2a+1, 2b+1}(x) - B_{2m-1}P_{2m-1}^{2a+1, 2b+1}(x)], \tag{3.3}$$

$$(g_{2m}^{(1)})^{1/2} \psi_{2m}^{(1)}(x) = \frac{w_{a+1,b+1}(x)}{A_{2m}}P_{2m-1}^{2a+1, 2b+1}(x) + \gamma_{2m-2}^{(2m)}\psi_{2m-2}^{(1)}(x), \tag{3.4}$$

$$(g_{2m}^{(1)})^{1/2} \phi_{2m}^{(1)}(x) = w_{a,b}(x)P_{2m}^{2a+1, 2b+1}(x), \tag{3.5}$$

with $(g_0^{(1)})^{1/2} \psi_0^{(1)}(x) = \int \epsilon(x - y)w_{a,b}(y) dy$ and

$$g_{2m}^{(1)} = g_{2m+1}^{(1)} = h_{2m}^{2a+1, 2b+1}, \quad m = 0, 1, \dots \tag{3.6}$$

For $\beta = 4$, with $\Phi_m^{(4)}(x)$ and $\Psi_m^{(4)}(x)$ satisfying conditions (i)–(iii), we have for $m \geq 1$,

$$(g_{2m}^{(4)})^{1/2} \phi_{2m+1}^{(4)}(x) = w_{a+1,b+1}(x)P_{2m-1}^{2a+1, 2b+1}(x), \tag{3.7}$$

$$(g_{2m}^{(4)})^{1/2} \psi_{2m+1}^{(4)}(x) = w_{a,b}(x)[A_{2m}P_{2m}^{2a+1, 2b+1}(x) - B_{2m-2}P_{2m-2}^{2a+1, 2b+1}(x)], \tag{3.8}$$

$$(g_{2m}^{(4)})^{1/2} \phi_{2m}^{(4)}(x) = -\frac{w_{a+1,b+1}(x)}{A_{2m-1}}P_{2m-2}^{2a+1, 2b+1}(x) + \gamma_{2m-3}^{(2m-1)}\phi_{2m-2}^{(4)}(x), \tag{3.9}$$

$$(g_{2m}^{(4)})^{1/2} \psi_{2m}^{(4)}(x) = -w_{a,b}(x)P_{2m-1}^{2a+1, 2b+1}(x), \tag{3.10}$$

with

$$g_{2m}^{(4)} = g_{2m+1}^{(4)} = h_{2m-1}^{2a+1, 2b+1}, \quad m = 1, 2, \dots, \tag{3.11}$$

where

$$\begin{aligned} \gamma_{j-2}^{(j)} &\equiv \gamma_j = \frac{(j+2a+2)(j+2b+2)}{(j+2)(j+2a+2b+4)}, \\ A_j &= -\frac{j(j+2a+2b+2)}{(2j+2a+2b+1)}, \quad B_j = -\frac{(j+2a+2)(j+2b+2)}{(2j+2a+2b+5)}, \end{aligned} \tag{3.12}$$

$$\frac{B_{j-1}}{A_j} h_{j-1}^{2a+1, 2b+1} = h_j^{2a+1, 2b+1}, \quad B_{-l} = 0, \quad l = 1, 2, \dots \tag{3.13}$$

$\phi_0^{(4)}(x)$ and $\phi_1^{(4)}(x)$ can be calculated using (6.1) and Gram–Schmidt method for SOP. Here we note that SOP for $\beta = 4$ is lower than that of $\beta = 1$ by an order 1.

3.2. Associated Laguerre SOP

Associated Laguerre ensembles of random matrices can and does play a significant role in describing real physical systems [42]. There exists a simple duality relation between the SOP $\Psi_m^{(1)}(x)$ and $\Phi_m^{(4)}(x)$, their derivatives and the normalization constant for $m \geq 1$,

$$\begin{aligned} \Psi_m^{(1)}(x) &= -\sigma_3 \Phi_m^{(4)}(x); & \Phi_m^{(1)}(x) &= -\sigma_3 \Psi_m^{(4)}(x); \\ g_m^{(4)} &= g_m^{(1)}, & \text{where} & \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{3.14}$$

is the Pauli matrix³.

We now present the results for the SOP corresponding to associated Laguerre weight. We use

$$2^{2a+1} w_a(x) w_{a+1}(x) = w_{2a+1}(2x), \quad w_{a+1}(x) = x w_a(x), \tag{3.15}$$

the latter vanishing at $x = 0$ for all $a > -1$.

For $\beta = 1$, with $(g_m^{(1)})^{1/2} \Phi_m^{(1)}(x) = w_a(x) \pi_m^{(1)}(x)$, we have for $m \geq 1$,

$$(g_{2m+1}^{(1)})^{1/2} \psi_{2m+1}^{(1)}(x) = 2^{a+3/2} w_{a+1}(x) L_{2m}^{2a+1}(2x) \tag{3.16}$$

$$(g_{2m+1}^{(1)})^{1/2} \phi_{2m+1}^{(1)}(x) = 2^{a+1/2} w_a(x) [A_{2m+1}^L L_{2m+1}^{2a+1}(2x) - B_{2m-1}^L L_{2m-1}^{2a+1}(2x)], \quad B_{-1}^L = 0, \tag{3.17}$$

$$(g_{2m}^{(1)})^{1/2} \psi_{2m}^{(1)}(x) = 2^{a+3/2} \frac{w_{a+1}(x)}{A_{2m}^L} L_{2m-1}^{2a+1}(x) + \gamma_{2m-2}^{(2m)} \psi_{2m-2}^{(1)}(x), \quad m \neq 0, \tag{3.18}$$

$$(g_{2m}^{(1)})^{1/2} \phi_{2m}^{(1)}(x) = 2^{a+1/2} w_a(x) L_{2m}^{2a+1}(2x), \tag{3.19}$$

$$g_{2m}^{(1)} = g_{2m+1}^{(1)} = h_{2m}^{2a+1}, \quad m = 0, 1, \dots, \tag{3.20}$$

where, $(g_0^{(1)})^{1/2} \psi_0^{(1)}(x) = 2^{a+1/2} \int_0^\infty \epsilon(x-y) w_a(y) dy$.

For $\beta = 4$, with $(g_m^{(4)})^{1/2} \Phi_m^{(4)}(x) = w_a(x) \pi_m^{(4)}(x)$, we have

$$(g_{2m+1}^{(4)})^{1/2} \phi_{2m+1}^{(4)}(x) = 2^{a+3/2} w_{a+1}(x) L_{2m}^{2a+1}(2x) \tag{3.21}$$

$$(g_{2m+1}^{(4)})^{1/2} \psi_{2m+1}^{(4)}(x) = 2^{a+1/2} w_a(x) [A_{2m+1}^L L_{2m+1}^{2a+1}(2x) - B_{2m-1}^L L_{2m-1}^{2a+1}(2x)] \tag{3.22}$$

³ Using (3.14) in (1.2) we can see that the partition functions corresponding to Laguerre weight also share a duality relation.

$$(g_{2m}^{(4)})^{1/2} \phi_{2m}^{(4)}(x) = -2^{a+3/2} \frac{w_{a+1}(x)}{A_{2m}^L} L_{2m-1}^{2a+1}(2x) + \gamma_{2m-3}^{(2m-1)} \phi_{2m-2}^{(4)}(x), \quad m \neq 0, \quad (3.23)$$

$$(g_{2m}^{(4)})^{1/2} \psi_{2m}^{(4)}(x) = -2^{a+1/2} w_a(x) L_{2m}^{2a+1}(2x) \quad (3.24)$$

$$g_{2m}^{(4)} = g_{2m+1}^{(4)} = h_{2m}^{2a+1}, \quad m = 0, 1, \dots, \quad (3.25)$$

where $(g_0^{(4)})^{1/2} \phi_0^{(4)}(x) \propto w_a(x)$.

Here $\gamma_{j-2}^{(j)} \equiv \gamma_j^L, A_j^L$ and B_j^L are given by

$$\gamma_j^L = \frac{(j + 2a + 2)}{(j + 2)}, \quad A_j^L = j, \quad B_j^L = j + 2a + 2, \quad (3.26)$$

$$\frac{B_{j-1}^L}{A_j^L} h_{j-1}^{2a+1} = h_j^{2a+1}, \quad j \geq 1. \quad (3.27)$$

Here, the identity

$$\frac{d}{dx} \{w_{a+1}(x) L_j^{(2a+1)}(2x)\} = \frac{1}{2} w_a(x) \{A_{j+1}^L L_{j+1}^{(2a+1)}(2x) - B_{j-1}^L L_{j-1}^{(2a+1)}(2x)\} \quad (3.28)$$

is used to derive (3.16) and (3.18) from (3.17) and (3.19) respectively.

3.3. Gaussian SOP

For Gaussian weight, with $w(x) = e^{-x^2/2}$ and $\beta = 1$, there exists a duality relation between the SOP $\Psi_m^{(1)}(x)$ and $\Phi_m^{(4)}(x)$ and their derivatives for $m \geq 1$, and is given by

$$(g_{2m+1}^{(1)})^{1/2} \psi_{2m+1}^{(1)}(x) = w(x) H_{2m}(x) \quad (3.29)$$

$$(g_{2m+1}^{(1)})^{1/2} \phi_{2m+1}^{(1)}(x) = w(x) [-(1/2) H_{2m+1}(x) + 2m H_{2m-1}(x)] \quad (3.30)$$

$$(g_{2m}^{(1)})^{1/2} \psi_{2m}^{(1)}(x) = -2w(x) H_{2m-1}(x) + 2(2m - 1) \psi_{2m-2}^{(1)}(x), \quad m \neq 0. \quad (3.31)$$

$$(g_{2m}^{(1)})^{1/2} \phi_{2m}^{(1)}(x) = w(x) H_{2m}(x) \quad (3.32)$$

$$g_{2m}^{(1)} = g_{2m+1}^{(1)} = h_{2m}, \quad m = 0, 1, \dots, \quad (3.33)$$

with $(g_0^{(1)})^{1/2} \psi_0^{(1)}(x) = \int_{-\infty}^{\infty} \epsilon(x - y) w(y) dy$ and $H_j(x)$ the ordinary Hermite polynomials.

For $\beta = 4$, we have

$$(g_{2m+1}^{(4)})^{1/2} \phi_{2m+1}^{(4)}(x) = w(x) H_{2m+1}(x) \quad (3.34)$$

$$(g_{2m+1}^{(4)})^{1/2} \psi_{2m+1}^{(4)}(x) = w(x) [-(1/2) H_{2m+2}(x) + (2m + 1) H_{2m}(x)] \quad (3.35)$$

$$(g_{2m}^{(4)})^{1/2} \phi_{2m}^{(4)}(x) = 2w(x) H_{2m}(x) + 4m \phi_{2m-2}^{(4)}(x) \quad (3.36)$$

$$(g_{2m}^{(4)})^{1/2} \psi_{2m}^{(4)}(x) = -w(x) H_{2m+1}(x) \quad (3.37)$$

$$g_{2m}^{(4)} = g_{2m+1}^{(4)} = h_{2m+1}, \quad m = 0, 1, \dots \quad (3.38)$$

Thus for the Gaussian weight, SOP for $\beta = 4$ is higher than that of $\beta = 1$ by an order 1. This is exactly the opposite of what we saw for Jacobi SOP.

3.4. Proof

The Jacobi SOP $\phi_j^{(4)}(x)$ and $\psi_j^{(1)}(x)$ can be written as

$$(g_j^{(4)})^{1/2} \phi_j^{(4)}(x) = \gamma_{j-1}^{(j-1)} w_{a+1,b+1} P_{j-2}^{2a+1,2b+1}(x) + \sum_{k=0}^{j-1} \gamma_{k-1}^{(j-1)} (g_k^{(4)})^{1/2} \phi_k^{(4)}(x),$$

$$j \geq 2, \quad \gamma_{j-1}^{(j-1)} \neq 0, \tag{3.39}$$

$$(g_j^{(1)})^{1/2} \psi_j^{(1)}(x) = \gamma_j^{(j)} w_{a+1,b+1} P_{j-1}^{2a+1,2b+1}(x) + \sum_{k=0}^{j-1} \gamma_k^{(j)} (g_k^{(1)})^{1/2} \psi_k^{(1)}(x),$$

$$j \geq 1, \quad \gamma_j^{(j)} \neq 0, \tag{3.40}$$

such that they satisfy conditions (i)–(iii). For example, $w_{a+1,b+1}(x)$ is used instead of $w_{a,b}(x)$ in order to satisfy condition (i). The order of the OP are fixed by (ii), while (iii) fixes $\gamma_k^{(j)}$.

Differentiating and using the identity

$$\frac{d}{dx} \{w_{a+1,b+1}(x) P_j^{2a+1,2b+1}(x)\} = w_{a,b}(x) \{A_{j+1} P_{j+1}^{2a+1,2b+1}(x) - B_{j-1} P_{j-1}^{2a+1,2b+1}(x)\}, \tag{3.41}$$

we get

$$(g_j^{(4)})^{1/2} \psi_j^{(4)}(x) = \gamma_{j-1}^{(j-1)} w_{a,b} [A_{j-1} P_{j-1}^{2a+1,2b+1}(x) - B_{j-3} P_{j-3}^{2a+1,2b+1}(x)]$$

$$+ \sum_{k=0}^{j-1} \gamma_{k-1}^{(j-1)} (g_k^{(4)})^{1/2} \psi_k^{(4)}(x) \tag{3.42}$$

$$(g_j^{(1)})^{1/2} \phi_j^{(1)}(x) = \gamma_j^{(j)} w_{a,b} [A_j P_j^{2a+1,2b+1}(x) - B_{j-2} P_{j-2}^{2a+1,2b+1}(x)]$$

$$+ \sum_{k=0}^{j-1} \gamma_k^{(j)} (g_k^{(1)})^{1/2} \phi_k^{(1)}(x). \tag{3.43}$$

In this paper we will give the proof for $\beta = 4$. The proof for $\beta = 1$ follows the same line of logic and can be found in [3, 4].

Using (3.1) and orthonormality of $P_j^{2a+1,2b+1}(x)$ w.r.t. the weight function $w_{2a+1,2b+1}(x)$, the scalar products

$$(\phi_{2m}^{(4)}(x), \psi_{2m-2k}^{(4)}(x)) = 0, \implies \gamma_{2m-2k}^{(2m-1)} = 0, \quad k = 1, 2, \dots \tag{3.44}$$

We also have

$$(\phi_{2m}^{(4)}(x), \psi_{2m-2k+1}^{(4)}(x)) = 0, \implies \gamma_{2m-2k-1}^{(2m-1)} = 0, \quad k = 2, 3, \dots, \tag{3.45}$$

$$(\phi_{2m+1}^{(4)}(x), \psi_{2m-2k}^{(4)}(x)) = 0, \implies \gamma_{2m-2k}^{(2m)} = 0, \quad k = 1, 2, \dots, \tag{3.46}$$

$$(\phi_{2m+1}^{(4)}(x), \psi_{2m-2k+1}^{(4)}(x)) = 0, \implies \gamma_{2m-2k-1}^{(2m)} = 0, \quad k = 1, 2, \dots \tag{3.47}$$

Since odd SOP is arbitrary to the addition of any multiple of the lower even SOP, we set $\gamma_{2m-1}^{(2m)} = 0$. Choosing $\gamma_{2m-1}^{(2m-1)} = -\frac{1}{A_{2m-1}}$ and $\gamma_{2m}^{(2m)} = 1$, we get

$$g_{2m}^{(4)} = g_{2m+1}^{(4)} = h_{2m-1}^{2a+1,2b+1}. \tag{3.48}$$

Here we have used (3.13). Finally,

$$(\psi_{2m}^{(4)}(x), \phi_{2m-1}^{(4)}(x)) = 0, \implies \gamma_{2m-3}^{(2m-1)} = \frac{B_{2m-3}}{A_{2m-1}}, \quad m \geq 2. \quad (3.49)$$

Thus we get (3.7)–(3.10).

To prove (3.21)–(3.24), we start with the expansion

$$(g_j^{(4)})^{1/2} \phi_j^{(4)}(x) = \gamma_{j-1}^{(j-1)} 2^{a+3/2} w_{a+1}(x) L_{j-1}^{2a+1}(2x) + \sum_{k=0}^{j-1} \gamma_{k-1}^{(j-1)} (g_k^{(4)})^{1/2} \phi_k^{(4)}(x),$$

$$j \geq 1, \quad \gamma_{j-1}^{(j-1)} \neq 0, \quad (3.50)$$

$$(g_j^{(1)})^{1/2} \psi_j^{(1)}(x) = \gamma_j^{(j)} 2^{a+3/2} w_{a+1}(x) L_{j-1}^{2a+1}(2x) + \sum_{k=0}^{j-1} \gamma_k^{(j)} (g_k^{(1)})^{1/2} \psi_k^{(1)}(x),$$

$$j \geq 1, \quad \gamma_j^{(j)} \neq 0. \quad (3.51)$$

Using (3.28), we get $\psi_j^{(4)}(x)$ and $\phi_j^{(1)}(x)$. Finally, we follow the same procedure and use (3.26)–(3.27) to obtain (3.21)–(3.25).

For Gaussian SOP, we expand

$$(g_j^{(4)})^{1/2} \phi_j^{(4)}(x) = \gamma_{j-1}^{(j-1)} w(x) H_j(x) + \sum_{k=0}^{j-1} \gamma_{k-1}^{(j-1)} (g_k^{(4)})^{1/2} \phi_k^{(4)}(x),$$

$$j = 0, 1, \dots, \quad \gamma_{j-1}^{(j-1)} \neq 0, \quad (3.52)$$

$$(g_j^{(1)})^{1/2} \psi_j^{(1)}(x) = \gamma_j^{(j)} w(x) H_{j-1}(x) + \sum_{k=0}^{j-1} \gamma_k^{(j)} (g_k^{(1)})^{1/2} \psi_k^{(1)}(x),$$

$$j \geq 1, \quad \gamma_j^{(j)} \neq 0, \quad (3.53)$$

and use the relation

$$\frac{d}{dx}(e^{-x^2/2} H_j(x)) = e^{-x^2/2} \left(-\frac{1}{2} H_{j+1}(x) + j H_{j-1}(x)\right) \quad (3.54)$$

to prove (3.34)–(3.37).

4. Classical orthogonal polynomials and some relevant formula

Orthogonal polynomials $P_j(x)$ of order j , associated with weight function $w(x)$ in the interval $[x_1, x_2]$ is defined as [19]

$$\int_{x_1}^{x_2} P_j(x) P_k(x) w(x) dx = h_j \delta_{j,k}, \quad P_j(x) = \sum_{l=0}^j k_l^{(j)} x^l, \quad j, k, l \in \mathbb{N}, \quad (4.1)$$

where h_j is the normalization constant and $k_j^{(j)}$ is the leading coefficient. They satisfy three-term recursion relation

$$x P_j(x) = Q_{j,j+1} P_{j+1}(x) + Q_{j,j} P_j(x) + Q_{j,j-1} P_{j-1}(x), \quad j = 0, 1, \dots, \quad (4.2)$$

where $Q_{j,k}$ is the recursion coefficient.

For classical weight functions (2.6)–(2.8), defined in the interval $[x_1, x_2]$, we have [19] the following table:

	Jacobi weight: [−1, 1]	Associated Laguerre weight: [0, ∞]	Gaussian weight: [−∞, ∞]
$P_0(x)$	1	1	1
$P_1(x)$	$\frac{1}{2}(a + b + 2)x + \frac{1}{2}(a - b)$	$-x + a + 1$	$2x$
(h_j)	$h_j^{a,b} = \left(\frac{2^{a+b+1}}{(2j+a+b+1)} \frac{\Gamma(j+a+1)\Gamma(j+b+1)}{\Gamma(j+1)\Gamma(j+a+b+1)} \right)$	$h_j^a = \frac{\Gamma(j+a+1)}{j!}$	$h_j = \pi^{1/2} 2^j j!$
$k_j^{(j)}$	$k_j^{a,b} = \frac{1}{2^j} \binom{2j+a+b}{j}$	$k_j^a = \frac{(-1)^j}{j!}$	2^j
$Q_{j-1,j}$	$\frac{2j(j+a+b)}{(2j+a+b)(2j+a+b-1)}$	$-j$	$1/2$
$Q_{j-1,j-1}$	$\frac{(b^2-a^2)}{(2j+a+b)(2j+a+b-2)}$	$2j + a - 1$	0
$Q_{j-1,j-2}$	$\frac{2(j+a-1)(j+b-1)}{(2j+a+b-1)(2j+a+b-2)}$	$-j + a - 1$	$j - 1$

4.1. Asymptotic formula

Here we give a brief summary of the asymptotic results [19] of OP with classical weight which will be useful in our analysis of the corresponding SOP.

Jacobi polynomial $P_j^{(a,b)}(x)$, for large j , arbitrary real a, b and fixed positive number ϵ , is written as

$$\begin{aligned}
 x &= \cos \theta, & \epsilon &\leq \theta \leq \pi - \epsilon, \\
 (h_j^{a,b})^{-1/2} (w_{a,b}(x))^{1/2} P_j^{(a,b)}(x) & & & \\
 &= \sqrt{\frac{2}{\pi \sin \theta}} \cos \left[\left(j + \frac{a+b+1}{2} \right) \theta - \left(a + \frac{1}{2} \right) \frac{\pi}{2} \right] + O(j^{-1}),
 \end{aligned} \tag{4.3}$$

where we have used [19] and $h_j^{a,b} \simeq 2^{(a+b)} j^{-1}$.

Associated Laguerre polynomial $L_j^a(x)$, for large j , has formula of the Plancherel–Rotach type [19] given below. For ‘ a ’ arbitrary and real, ϵ a fixed positive number, we have for

$$\begin{aligned}
 x &= (4j + 2a + 2) \cos^2 \theta, & \epsilon &\leq \theta \leq \pi/2 - \epsilon j^{-\frac{1}{2}}, \\
 e^{-x/2} L_j^{(a)}(x) &= \frac{(-1)^j}{\sqrt{2\pi j \sin \theta \cos \theta}} \left(\frac{j}{x} \right)^{a/2} \left\{ \sin \left[(j + (a + 1)/2)(\sin 2\theta - 2\theta) + \frac{3\pi}{4} \right] \right\} \\
 &+ (jx)^{-\frac{1}{2}} O(1),
 \end{aligned} \tag{4.4}$$

where $h_j^{(a)} \simeq j^a$. Unlike the Jacobi case, for given x, θ depends on j ; for example $\theta_j - \theta_{j\mp 1} \simeq \pm(2j \tan \theta_j)^{-1}$. Then with $\theta \equiv \theta_j$, we can also write

$$\begin{aligned}
 e^{-x/2} L_{j\mp \Delta j}^{(a)}(x) &= \frac{(-1)^{j-1}}{\sqrt{2\pi j \sin \theta \cos \theta}} \left(\frac{j}{x} \right)^{a/2} \\
 &\times \left\{ \sin \left[(j + (a + 1)/2)(\sin 2\theta - 2\theta) \pm 2\theta \Delta j + \frac{3\pi}{4} \right] \right\} + (jx)^{-\frac{1}{2}} O(1).
 \end{aligned} \tag{4.5}$$

Equation (4.5) will be useful in deriving the asymptotic formulae for SOP.

Finally, Hermite polynomial $H_j(x)$, for large j , has formula of the Plancherel–Rotach type given below. For ϵ a fixed positive number, we have

$$\begin{aligned}
 x &= (2j + 1)^{\frac{1}{2}} \cos \theta, & \epsilon \leq \theta \leq \pi - \epsilon, \\
 e^{-x^2/2} H_j(x) &= \frac{(h_j)^{1/2}}{\sqrt{\pi \sin \theta}} \left(\frac{2}{j}\right)^{1/4} \sin \left[(j/2 + 1/4)(\sin 2\theta - 2\theta) + \frac{3\pi}{4} \right] + O(j^{-1}).
 \end{aligned}
 \tag{4.6}$$

Unlike the Jacobi case, for given x , θ depends on j ; for example $\theta_j - \theta_{j\mp 1} \simeq \pm(2j \tan \theta_j)^{-1}$. Then with $\theta \equiv \theta_j$, we can also write

$$e^{-x^2/2} H_{j\mp 1}(x) = \frac{(h_{j\mp 1})^{1/2}}{\sqrt{\pi \sin \theta}} \left(\frac{2}{j}\right)^{1/4} \sin \left[(j/2 + 1/4)(\sin 2\theta - 2\theta) \pm \theta + \frac{3\pi}{4} \right] + O(j^{-1}),
 \tag{4.7}$$

where we have used again $\theta_j - \theta_{j\mp 1} \simeq \pm(2j \tan \theta_j)^{-1}$.

5. Universality for orthogonal ensembles

In this section, we use GCD formula for orthogonal ensembles of random matrices (2.32) along with the asymptotic results of SOP [3, 4] to prove that with proper scaling, the kernel function $(S_{2N}^{(1)}(x, y)/S_{2N}^{(1)}(x, x))$ and hence the correlation function for Jacobi orthogonal and associated Laguerre orthogonal ensemble is stationary and universal.

5.1. Jacobi SOP

The asymptotic results for the SOP are derived using (4.3) in (3.2)–(3.5). Here, $A_j \approx B_j \approx -j/2$ and $\gamma_j \approx 1$ for large j . For

$$\begin{aligned}
 x &= \cos \theta, & \epsilon \leq \theta \leq \pi - \epsilon, \\
 \psi_{2m+1}^{(1)}(x) &= \sqrt{\frac{2 \sin \theta}{\pi}} \cos \left[\left(2m + a + b + \frac{3}{2}\right) \theta - \left(2a + \frac{3}{2}\right) \frac{\pi}{2} \right] + O(2m)^{-1},
 \end{aligned}
 \tag{5.1}$$

$$\psi_{2m}^{(1)}(x) = -\frac{1}{m\sqrt{2\pi \sin \theta}} \left[\sin \left[\left(2m + a + b + \frac{3}{2}\right) \theta - \left(2a + \frac{3}{2}\right) \frac{\pi}{2} \right] + O(1) + O(2m)^{-1} \right].
 \tag{5.2}$$

$$\phi_{2m+1}^{(1)}(x) = 2m \sqrt{\frac{2}{\pi \sin \theta}} \left[\sin \left[\left(2m + a + b + \frac{3}{2}\right) \theta - \left(2a + \frac{3}{2}\right) \frac{\pi}{2} \right] + O(2m)^{-1} \right],
 \tag{5.3}$$

$$\phi_{2m}^{(1)}(x) = \sqrt{\frac{2}{\pi \sin^3 \theta}} \cos \left[\left(2m + a + b + \frac{3}{2}\right) \theta - \left(2a + \frac{3}{2}\right) \frac{\pi}{2} \right] + O(2m)^{-1}.
 \tag{5.4}$$

Equations (5.1), (5.3) and (5.4) are obtained by using (4.3) in (3.2), (3.3) and (3.5). Equation (5.2) is obtained by partial integration of (5.4). Here, we note that by directly differentiating $\psi^{(1)}(x)$, we can get $\phi^{(1)}(x)$ to the leading order, thereby confirming our result.

5.2. Universality in Jacobi orthogonal ensemble

In this subsection, we calculate the level density of Jacobi orthogonal ensemble and show that in the bulk of the spectrum, the scaled or ‘unfolded’ [28] kernel function $(S_{2N}^{(1)}(x, y)/S_{2N}^{(4)}(x, x))$ is stationary and universal.

To study the kernel function (2.32), we calculate $P_{2N-1,2N}^{(1)}, R_{2N-1,2N+1}^{(1)}, R_{2N-1,2N}^{(1)}, R_{2N-2,2N}^{(1)}$. We expand

$$(1 - x^2)\phi_{2m+1}^{(1)}(x) = \sum_{j=2m-2}^{2m+2} P_{2m+1,j}^{(1)}\psi_j^{(1)}(x), \tag{5.5}$$

$$x(1 - x^2)\phi_{2m+1}^{(1)} = \sum_{j=2m-2}^{2m+3} R_{2m+1,j}^{(1)}\psi_j^{(1)}(x), \quad x(1 - x^2)\phi_{2m}^{(1)}(x) = \sum_{2m-2}^{2m+2} R_{2m,j}^{(1)}\psi_j^{(1)}(x). \tag{5.6}$$

We get

$$P_{2m+1,2m+2}^{(1)} = \sqrt{\frac{g_{2m+2}^{(1)}(2m+1)(2m+2)(2m+2a+2b+3)(2m+2a+2b+4)}{g_{2m}^{(1)}(4m+2a+2b+3)(4m+2a+2b+5)}}, \tag{5.7}$$

$$R_{2m+1,2m+3}^{(1)} = -2\sqrt{\frac{g_{2m+2}^{(1)}(2m+1)(2m+2)(2m+2a+2b+3)(2m+2a+2b+4)}{g_{2m}^{(1)}(4m+2a+2b+3)(4m+2a+2b+5)(4m+2a+2b+6)}}, \tag{5.8}$$

$$R_{2m+1,2m+2}^{(1)} = \sqrt{\frac{g_{2m+2}^{(1)}[(2b+1)^2 - (2a+1)^2](2m+1)(2m+2)(2m+2a+2b+3)(2m+2a+2b+4)}{g_{2m}^{(1)}(4m+2a+2b+3)(4m+2a+2b+4)(4m+2a+2b+5)(4m+2a+2b+6)}},$$

$$= 0, \quad \text{for } a = b, \tag{5.9}$$

$$R_{2m,2m+2}^{(1)} = -2\sqrt{\frac{g_{2m+2}^{(1)}(2m+1)(2m+2)(2m+2a+2b+3)(2m+2a+2b+4)}{g_{2m}^{(1)}(4m+2a+2b+3)(4m+2a+2b+4)(4m+2a+2b+5)}}. \tag{5.10}$$

For $m = N - 1$, large N , we have $(g_{2N}^{(4)}/g_{2N-2}^{(4)}) \simeq 1$ and

$$P_{2N-1,2N}^{(1)} \sim N^2 + O(N),$$

$$R_{2N-1,2N+1}^{(1)} \sim -\frac{N}{2} + O(1), \quad R_{2N-1,2N}^{(1)} \sim (b - a)O(1), \quad R_{2N-2,2N}^{(1)} \sim -\frac{N}{2} + O(1). \tag{5.11}$$

Finally using (5.1) and (5.2), defined in the θ interval $[\epsilon, \pi - \epsilon]$, and (5.11) in the GCD formula (2.32), we get for

$$x = \cos \theta, \quad y = x + \Delta x = \cos(\theta + \Delta\theta), \quad x - y \simeq \Delta\theta \sin \theta,$$

$$(x - y)(1 - x^2)S_{2N}^{(1)}(x, y) = R_{2N-2,2N}^{(1)}[\psi_{2N}^{(1)}(x)\psi_{2N-1}^{(1)}(y) - \psi_{2N}^{(1)}(y)\psi_{2N-1}^{(1)}(x)]$$

$$+ R_{2N-1,2N+1}^{(1)}[\psi_{2N-2}^{(1)}(x)\psi_{2N+1}^{(1)}(y) - \psi_{2N-2}^{(1)}(y)\psi_{2N+1}^{(1)}(x)]$$

$$+ (R_{2N-1,2N}^{(1)} - xP_{2N-1,2N}^{(1)})[\psi_{2N-2}^{(1)}(x)\psi_{2N}^{(1)}(y) - \psi_{2N-2}^{(1)}(y)\psi_{2N}^{(1)}(x)]$$

$$= \frac{1}{2\pi}[\sin(f_{2N}(\theta)) \cos(f_{2N-2}(\theta + \Delta\theta)) - \sin(f_{2N}(\theta + \Delta\theta)) \cos(f_{2N-2}(\theta))]$$

$$+ \frac{1}{2\pi}[\sin(f_{2N-2}(\theta)) \cos(f_{2N}(\theta + \Delta\theta)) - \sin(f_{2N-2}(\theta + \Delta\theta)) \cos(f_{2N}(\theta))]$$

$$- \frac{\cos \theta}{2\pi \sin \theta}[\sin(f_{2N-2}(\theta)) \sin(f_{2N}(\theta + \Delta\theta))$$

$$- \sin(f_{2N-2}(\theta + \Delta\theta)) \sin(f_{2N}(\theta))],$$

where, in the second step, we have dropped $O(1/N^2)$ term. Thus we get

$$\begin{aligned}
 (x - y)(1 - x^2)S_{2N}^{(1)}(x, y) &= \frac{1}{2\pi} [\sin[f_{2N}(\theta) - (f_{2N-2}(\theta + \Delta\theta))] \\
 &\quad - \sin[f_{2N-2}(\theta) - f_{2N}(\theta + \Delta\theta)] - \frac{\cos\theta \sin(2\theta)}{2\pi \sin\theta} [\sin[\Delta\theta f'_{2N}(\theta)]] \\
 &= -\frac{1}{2\pi} [\sin[\Delta\theta f'_{2N}(\theta) - 2\theta] + \sin[\Delta\theta f'_{2N}(\theta) + 2\theta]] \\
 &\quad + \frac{\cos\theta \sin(2\theta)}{2\pi \sin\theta} [\sin[\Delta\theta f'_{2N}(\theta)]] \\
 &\simeq -\frac{1}{\pi} \sin[2N\Delta\theta][\cos(2\theta) - \cos^2\theta], \tag{5.12}
 \end{aligned}$$

which gives us finally

$$S_{2N}^{(1)}(x, y) = \frac{\sin(2N\Delta\theta)}{\pi \Delta\theta \sin\theta}, \tag{5.13}$$

$$= \frac{\sin(2N(1 - x^2)^{-1/2} \Delta x)}{\pi \Delta x}, \quad |x| < 1. \tag{5.14}$$

With $x \rightarrow y$, we get the level density $\rho(x)$

$$S_{2N}^{(1)}(x, x) := \rho(x) = \frac{2N}{\pi \sqrt{1 - x^2}}, \quad |x| < 1. \tag{5.15}$$

With $\Delta x \rightarrow 0$, i.e. in the bulk of the spectrum, we get the ‘universal’ sine-kernel

$$\frac{S_{2N}^{(1)}(x, y)}{S_{2N}^{(1)}(x, x)} = \frac{\sin \pi r}{\pi r}, \quad r = \rho(x)\Delta x. \tag{5.16}$$

5.3. Associated Laguerre SOP

To obtain the asymptotic properties of associated Laguerre SOP, we use the relations $2^{a+1/2}\sqrt{x}w_a(x) = \sqrt{w_{2a+1}(2x)}$ and $2^{a+1/2}w_{a+1}(x) = \sqrt{x}\sqrt{w_{2a+1}(2x)}$. Replacing this in (3.16)–(3.19) and using (4.4) we get the asymptotic formula.

For arbitrary a and ϵ a fixed positive number,

$$2x = (8m + 4a + 4) \cos^2 \theta, \quad \epsilon \leq \theta \leq \pi/2 - \epsilon m^{-1/2}, \quad \theta \equiv \theta_{2m}, \tag{5.17}$$

$$\phi_{2m}^{(1)}(x) = \frac{1}{4m\sqrt{\pi \sin\theta \cos^3\theta}} \left[\sin(f_{2m}(\theta)) + \frac{O(1)}{\sqrt{2mx}} \right],$$

$$\psi_{2m+1}^{(1)}(x) = \frac{2}{\sqrt{\pi \tan\theta}} \left[\sin(f_{2m}(\theta)) + \frac{O(1)}{\sqrt{2mx}} \right], \tag{5.18}$$

$$\psi_{2m}^{(1)}(x) = -\frac{1}{4m\sqrt{\pi \sin^3\theta \cos\theta}} \left[\cos(f_{2m}(\theta)) + O(1) + \frac{O(1)}{\sqrt{2mx}} \right], \tag{5.19}$$

$$\phi_{2m+1}^{(1)}(x) = 2\sqrt{\frac{\tan\theta}{\pi}} \left[\cos(f_{2m}(\theta)) + \frac{O(1)}{\sqrt{2mx}} \right], \tag{5.20}$$

with

$$\phi_{2m\pm 2}^{(1)}(x) = \frac{1}{4m\sqrt{\pi \sin\theta \cos^3\theta}} \left[\sin(f_{2m}(\theta) \mp 4\theta) + \frac{O(1)}{\sqrt{2mx}} \right]. \tag{5.21}$$

Similar relations hold for (5.18)–(5.20) as $m \rightarrow m \pm 1$. Here

$$f_{2m}(\theta) = (2m + a + 1)(\sin 2\theta - 2\theta) + \frac{3\pi}{4}. \tag{5.22}$$

5.4. Universality in associated Laguerre orthogonal ensemble

In this subsection, we calculate the level density of associated Laguerre orthogonal ensemble and show that in the bulk of the spectrum, the scaled or ‘unfolded’ [28] kernel function $(S_{2N}^{(1)}(x, y)/S_{2N}^{(1)}(x, x))$ is stationary and universal.

To calculate the kernel function (2.32), we need $R_{2N-1,2N+1}^{(1)}, R_{2N-1,2N}^{(1)}, R_{2N-2,2N}^{(1)}$ and $P_{2N-1,2N}^{(1)}$. For this, we expand

$$x\phi_{2m+1}^{(1)}(x) = \sum_{j=2m-2}^{2m+2} P_{2m+1,j}^{(1)}\psi_j^{(1)}(x), \tag{5.23}$$

$$x^2\phi_{2m}^{(1)}(x) = \sum_{j=2m-2}^{2m+2} R_{2m,j}^{(1)}\psi_j^{(1)}(x), \quad x^2\phi_{2m+1}^{(1)}(x) = \sum_{j=2m-2}^{2m+3} R_{2m+1,j}^{(1)}\psi_j^{(1)}(x), \tag{5.24}$$

Using (3.17) in (5.23), we get

$$P_{2m+1,2m+2}^{(1)} = \frac{1}{2} \sqrt{\frac{g_{2m+2}^{(1)}}{g_{2m}^{(1)}}(2m+1)(2m+2)}. \tag{5.25}$$

Using (3.19) and (3.17) in (5.24), we get

$$R_{2m,2m+2}^{(1)} = R_{2m+1,2m+3}^{(1)} = -\frac{1}{2} \sqrt{\frac{g_{2m+2}^{(1)}}{g_{2m}^{(1)}}(m+1)(2m+1)}, \tag{5.26}$$

and

$$R_{2m+1,2m+2}^{(1)} = \frac{1}{2} \sqrt{\frac{g_{2m+2}^{(1)}}{g_{2m}^{(1)}}(m+1)(2m+1)(4m+2a+4)}. \tag{5.27}$$

For $m = N - 1$, large N , we have $(g_{2N}^{(1)}/g_{2N-2}^{(1)}) \simeq 1$ and

$$\begin{aligned} P_{2N-1,2N}^{(1)} &\sim 2N^2 + O(N), & R_{2N-1,2N+1}^{(1)} &\sim -N^2 + O(N), \\ R_{2N-1,2N}^{(1)} &\sim 4N^3 + O(N^2), & R_{2N-2,2N}^{(1)} &\sim -N^2 + O(N). \end{aligned} \tag{5.28}$$

Finally using (5.18), (5.19) and (5.21), defined in the θ interval $[\epsilon, \pi/2 - \epsilon N^{-1/2}]$, and (5.28) in the GCD formula (2.32), we get for

$$\begin{aligned} x &= (4N + 2a + 2) \cos^2 \theta, & y &= x + \Delta x, \\ x(x - y)S_{2N}^{(1)}(x, y) &= R_{2N-2,2N}^{(1)}[\psi_{2N}^{(1)}(x)\psi_{2N-1}^{(1)}(y) - \psi_{2N}^{(1)}(y)\psi_{2N-1}^{(1)}(x)] \\ &\quad + R_{2N-1,2N+1}^{(1)}[\psi_{2N-2}^{(1)}(x)\psi_{2N+1}^{(1)}(y) - \psi_{2N-2}^{(1)}(y)\psi_{2N+1}^{(1)}(x)] \\ &\quad + (R_{2N-1,2N}^{(1)} - xP_{2N-1,2N}^{(1)})[\psi_{2N-2}^{(1)}(x)\psi_{2N}^{(1)}(y) - \psi_{2N-2}^{(1)}(y)\psi_{2N}^{(1)}(x)] \\ &= \frac{N}{2\pi \sin^2 \theta} [\sin(f_{2N-2}(\theta + \Delta\theta) - f_{2N}(\theta)) + \sin(f_{2N}(\theta + \Delta\theta) - (f_{2N-2}(\theta)))] \\ &\quad - \left[\frac{N}{2\pi \sin^2 \theta \tan 2\theta} \right] [\cos(f_{2N-2}(\theta)) \cos(f_{2N}(\theta + \Delta\theta)) \\ &\quad - \cos(f_{2N-2}(\theta + \Delta\theta)) \cos(f_{2N}(\theta))] \\ &= \frac{N \cos 4\theta}{\pi \sin^2 \theta} \sin\left(\Delta\theta \frac{\partial f_{2N}(\theta)}{\partial \theta}\right) - \left[\frac{N \sin 4\theta}{2\pi \sin^2 \theta \tan 2\theta} \right] \sin\left(\Delta\theta \frac{\partial f_{2N}(\theta)}{\partial \theta}\right). \end{aligned} \tag{5.29}$$

Here, we have neglected the oscillatory term of $O(1)$ arising from the even function. Finally, we get

$$\begin{aligned} (x - y)S_{2N}^{(1)}(x, y) &= \frac{1}{\pi \sin^2 2\theta} (\cos 4\theta - \cos^2 2\theta) \left[\sin \left(\Delta\theta \frac{\partial f_{2N}(\theta)}{\partial \theta} \right) \right] \\ &= \frac{1}{\pi} \sin[8N \sin^2 \theta \Delta\theta]. \end{aligned} \tag{5.30}$$

Combining all, we get

$$S_{2N}^{(1)}(x, y) = \frac{\sin(x^{-1/2} \sqrt{(4N - x)} \Delta x)}{\pi \Delta x}, \quad 0 < x < 4N. \tag{5.31}$$

With $x \rightarrow y$, we get the level density

$$S_{2N}^{(1)}(x, x) = \frac{1}{\pi} \sqrt{\frac{4N - x}{x}}, \quad 0 < x < 4N. \tag{5.32}$$

With $\Delta x \rightarrow 0$, (i.e. in the bulk of the spectrum) and $r = \Delta x S_{2N}^{(1)}(x, x)$, we get the ‘universal’ sine-kernel (5.16).

6. Universality for symplectic ensembles

Before we prove universality of the eigenvalue correlation for symplectic ensembles, we would clarify some of the confusing notations related to the corresponding SOP.

SOP corresponding to symplectic ensembles of random matrices can be defined in an interval $[x_1, x_2]$ as

$$\begin{aligned} &\int_{x_1}^{x_2} g_j^{-1} [\pi_j^{(4)}(x) \pi_k^{(4)'}(x) - \pi_k^{(4)}(x) \pi_j^{(4)'}(x)] w(x) dx \\ &= \int_{x_1}^{x_2} [\phi_j^{(4)}(x) \psi_k^{(4)}(x) - \phi_k^{(4)}(x) \psi_j^{(4)}(x)] dx = Z_{j,k}, \end{aligned} \tag{6.1}$$

where

$$\phi_j^{(4)}(x) = (g_j)^{-\frac{1}{2}} w(x)^{1/2} \pi_j^{(4)}(x), \quad \psi_j^{(4)}(x) = \frac{d}{dx} \phi_j^{(4)}(x). \tag{6.2}$$

Here, $\pi_j^{(4)}(x)$ are SOP defined with respect to $w(x)$. This definition is used in [3, 4] and will also be used in this paper to study the statistical properties of the symplectic ensembles.

An alternative definition is if we write

$$\phi_j^{(4)}(x) = (g_j)^{-\frac{1}{2}} w(x) \pi_j^{(4)}(x), \quad \psi_j^{(4)}(x) = \frac{d}{dx} \phi_j^{(4)}(x), \tag{6.3}$$

such that

$$\begin{aligned} &\frac{1}{2} \int_{x_1}^{x_2} g_j^{-1} [\pi_j^{(4)}(x) \pi_k^{(4)'}(x) - \pi_k^{(4)}(x) \pi_j^{(4)'}(x)] w^2(x) dx \\ &= \frac{1}{2} \int_{x_1}^{x_2} [\phi_j^{(4)}(x) \psi_k^{(4)}(x) - \phi_k^{(4)}(x) \psi_j^{(4)}(x)] dx \\ &= \int_{x_1}^{x_2} \phi_j^{(4)}(x) \psi_k^{(4)}(x) dx \\ &= Z_{j,k}. \end{aligned} \tag{6.4}$$

Here $\phi_j^{(4)}(x_1) = \phi_j^{(4)}(x_2) = 0$. The SOP $\pi_j^{(4)}(x)$ in this definition is defined with respect to $w^2(x)$. This definition is used in [5, 6] and is used to prove duality between SOP corresponding

to orthogonal and symplectic ensembles of random matrices. Also, this definition differs from that of [3, 4, 28] for $\beta = 4$ by a factor 2, which is incorporated in the normalization constant.

We will use (6.1) and the GCD formula to prove universality. Here we would like to mention that our GCD results are valid for both these definitions with some minor difference in $\overline{R}^{(4)}(x)$.

6.1. Jacobi SOP

We consider SOP (6.2) defined with respect to the Jacobi weight (2.6). As shown in [3, 4], $\pi_j^{(4)}(x) \equiv \pi_j(x)$ and $\pi'_j(x)$ can be written compactly in terms of Jacobi OP $P_j^{a,b}(x)$,

$$\pi'_{2m+1}(x) = P_{2m}^{a,b}(x), \quad m = 0, 1, \dots, \tag{6.5}$$

$$\pi'_{2m}(x) = P_{2m-1}^{a,b}(x) + \eta_{2m}\pi'_{2m-2}(x), \quad m = 1, 2, \dots, \quad \pi'_0(x) = 0, \tag{6.6}$$

where η_{2m} is a constant, given in equation (6.11). On integration, we find the polynomials

$$\pi_{2m+1}(x) = \frac{2}{(2m+a+b)} [D_{2m+1}P_{2m+1}^{a,b}(x) + E_{2m+1}P_{2m}^{a,b}(x) + F_{2m+1}P_{2m-1}^{a,b}(x)], \tag{6.7}$$

$$m = 0, 1, \dots,$$

$$\pi_{2m}(x) = \frac{2}{(2m+a+b-1)} [D_{2m}P_{2m}^{a,b}(x) + E_{2m}P_{2m-1}^{a,b}(x) + F_{2m}P_{2m-2}^{a,b}(x)] + \eta_{2m}\pi_{2m-2}(x), \tag{6.8}$$

$$m = 0, 1, \dots$$

Here, $P_j^{a,b}(x) = 0$ for negative j . In (6.7) and (6.8) we have used the indefinite integral

$$\frac{1}{2}(j+a+b) \int P_j^{a,b}(x) dx = P_{j+1}^{a-1,b-1}(x), \tag{6.9}$$

$$= D_{j+1}P_{j+1}^{a,b}(x) + E_{j+1}P_j^{a,b}(x) + F_{j+1}P_{j-1}^{a,b}(x).$$

The integration constants have been put equal to zero because of skew-orthogonality with $\pi_1(x)$. The constants D_j, E_j, F_j, η_j and $g_j^{(4)}$ are given by

$$D_j = \frac{(j+a+b)(j+a+b-1)}{(2j+a+b)(2j+a+b-1)}, \quad E_j = \frac{(a-b)(j+a+b-1)}{(2j+a+b)(2j+a+b-2)}, \tag{6.10}$$

$$F_j = -\frac{(j+a-1)(j+b-1)}{(2j+a+b-1)(2j+a+b-2)}, \quad \eta_j = \frac{(j+a-1)(j+b-1)(2j+a+b-5)}{(j-1)(j+a+b-1)(2j+a+b-1)},$$

$$g_{2m}^{(4)} = g_{2m+1}^{(4)} = \frac{2h_{2m}^{a,b}}{4m+a+b-1}, \tag{6.11}$$

$$= \frac{2^{a+b+2}\Gamma(2m+a+1)\Gamma(2m+b+1)}{(4m+a+b+1)(4m+a+b-1)\Gamma(2m+1)\Gamma(2m+a+b+1)}.$$

For large j and large m ,

$$D_j = -F_j = \frac{1}{4} + O(j^{-1}), \quad E_j = (a-b) \left[\frac{1}{4j} + O(j^{-2}) \right], \tag{6.12}$$

$$\eta_j = 1 + O(j^{-1}), \quad g_{2m}^{(4)} = \frac{2^{a+b}}{4m^2} + O(m^{-3}), \tag{6.13}$$

and in the same approximation,

$$\begin{aligned} \pi_{2m+1}(x) &= \frac{1}{4m} [P_{2m+1}^{a,b}(x) - P_{2m-1}^{a,b}(x)], \\ \pi_{2m}(x) &= \frac{1}{4m} [P_{2m}^{a,b}(x) + 2^{(a+b)/2} (w_{a,b}(x))^{-1/2}]. \end{aligned} \tag{6.14}$$

Here in (6.14), the non-polynomial term on the right-hand side is the large- m approximation for the lower-order terms in the series in (6.8) and has been verified numerically.

For large m , we use (6.14) and the asymptotic formula for Jacobi OP (4.3) to obtain asymptotic formula for Jacobi SOP. For a, b arbitrary and real, ϵ a fixed positive number, we have for

$$x = \cos \theta, \quad \epsilon \leq \theta \leq \pi - \epsilon, \tag{6.15}$$

$$(g_{2m}^{(4)})^{-1/2} \sqrt{w_{a,b}(x)} \pi_{2m+1}(x) := \phi_{2m+1}^{(4)}(x) = -\sqrt{\frac{\sin \theta}{\pi m}} \sin[f_{2m}(\theta)] + O(m^{-3/2}),$$

$$(g_{2m}^{(4)})^{-1/2} \sqrt{w_{a,b}(x)} \pi_{2m}(x) := \phi_{2m}^{(4)}(x) = \frac{1}{2} \left[\frac{1}{\sqrt{\pi m \sin \theta}} \cos[f_{2m}(\theta)] + 1 \right] + O(m^{-3/2}), \tag{6.16}$$

$$\psi_{2m+1}^{(4)}(x) = 2\sqrt{\frac{m}{\pi \sin \theta}} \cos[f_{2m}(\theta)] + O(m^{-1/2}), \tag{6.17}$$

$$\psi_{2m}^{(4)}(x) = \sqrt{\frac{m}{\pi \sin^3 \theta}} [\sin[f_{2m}(\theta)] + 1] + O(m^{-1/2}), \tag{6.18}$$

where

$$f_{2m}(\theta) = \left(2m + \frac{a+b+1}{2}\right) \theta - \left(a + \frac{1}{2}\right) \frac{\pi}{2}. \tag{6.19}$$

Equation (6.17) is derived from (6.5) and (6.18) is obtained by differentiating (6.16). Here, we would like to mention that to calculate level density and two-point correlation function for the Jacobi symplectic ensemble, (6.17) and (6.18) are not needed. However, they are important to define the SOP and hence included for completeness.

6.2. Level density and two-point correlation for Jacobi symplectic ensemble

In this subsection, we calculate the level density of Jacobi symplectic ensemble and show that in the bulk of the spectrum, the scaled or ‘unfolded’ [28] kernel function $(S_{2N}^{(4)}(x, y) / S_{2N}^{(4)}(y, y))$ is stationary and universal.

As suggested in equation (2.31), to obtain the kernel function, we need to calculate $R_{2N-1, 2N+1}^{(4)}, R_{2N-1, 2N}^{(4)}, R_{2N-2, 2N}^{(4)}$ and $P_{2N-1, 2N}^{(4)}$. For this, we use (2.18),

$$(1-x^2) \frac{d}{dx} \phi_{2m+1}^{(4)}(x) = \sum_{j=2m-2}^{2m+2} P_{2m+1, j}^{(4)} \phi_j^{(4)}(x), \tag{6.20}$$

to get $P_{2m+1, 2m+2}^{(4)}$. It is given by

$$P_{2m+1, 2m+2}^{(4)} = -\sqrt{\frac{g_{2m+2}^{(4)}}{g_{2m}^{(4)}}} \frac{(2m+a+b+1) Q_{2m, 2m+1} Q_{2m+1, 2m+2}}{2D_{2m+2}}, \tag{6.21}$$

where $Q_{j,k}$ and D_j are given in the table and (6.10) respectively.

Similarly, we use (2.19),

$$\begin{aligned}
 x(1-x^2)\frac{d}{dx}\phi_{2m}^{(4)}(x) &= \sum_{j=2m-2}^{2m+2} R_{2m,j}^{(4)}\phi_j^{(4)}(x), \\
 x(1-x^2)\frac{d}{dx}\phi_{2m+1}^{(4)}(x) &= \sum_{j=2m-2}^{2m+3} R_{2m+1,j}^{(4)}\phi_j^{(4)}(x),
 \end{aligned}
 \tag{6.22}$$

to get $R_{2m,2m+2}^{(4)}$, $R_{2m+1,2m+3}^{(4)}$ and $R_{2m+1,2m+2}^{(4)}$. For large m , we get to the leading order

$$\begin{aligned}
 R_{2m+1,2m+2}^{(4)} &\simeq \sqrt{\frac{g_{2m+2}^{(4)}}{g_{2m}^{(4)}}} \left[\frac{(2m+a+b+1)E_{2m+3}Q_{2m,2m+1}Q_{2m+1,2m+2}Q_{2m+2,2m+3}}{2D_{2m+3}D_{2m+2}} \right. \\
 &\quad \left. - (a-b)\frac{(2m+a+b+1)}{2(2m+a+b)}\frac{D_{2m+1}}{D_{2m+2}}Q_{2m+1,2m+2} \right] + (a-b)O(1/m)
 \end{aligned}
 \tag{6.23}$$

$$= 0 \quad \text{for} \quad a = b,
 \tag{6.24}$$

$$R_{2m+1,2m+3}^{(4)} \simeq -\sqrt{\frac{g_{2m+2}^{(4)}}{g_{2m}^{(4)}}} \frac{(2m+a+b+2)Q_{2m,2m+1}Q_{2m+1,2m+2}Q_{2m+2,2m+3}}{2D_{2m+3}} + O(1),
 \tag{6.25}$$

$$R_{2m,2m+2}^{(4)} \simeq -\sqrt{\frac{g_{2m+2}^{(4)}}{g_{2m}^{(4)}}} \frac{(2m+a+b+1)Q_{2m-1,2m}Q_{2m,2m+1}Q_{2m+1,2m+2}}{2D_{2m+2}} + O(1),
 \tag{6.26}$$

where $Q_{j,k}$, D_j and E_j are given in the table and (6.10) respectively.

For large j , we use $Q_{j,j+1} \simeq 1/2$. Also for $m = N-1$, large N , we have $(g_{2N}^{(4)}/g_{2N-2}^{(4)}) \simeq 1$ and

$$\begin{aligned}
 P_{2N-1,2N}^{(4)} &\sim -N + O(1), & R_{2N-1,2N+1}^{(4)} &\sim -\frac{N}{2} + O(1), \\
 R_{2N-1,2N}^{(4)} &\sim \frac{(b-a)}{4} + (b-a)O(N^{-1}), & R_{2N-2,2N}^{(4)} &\sim -\frac{N}{2} + O(1).
 \end{aligned}
 \tag{6.27}$$

Finally using (6.15) and (6.16), defined in the θ interval $[\epsilon, \pi - \epsilon]$, and (6.27) in the GCD formula (2.31), we get for

$$\begin{aligned}
 y = \cos \theta, \quad x = y + \Delta y = \cos(\theta + \Delta\theta), \quad y - x &\simeq \Delta\theta \sin \theta, \\
 (y-x)(1-y^2)S_{2N}^{(4)}(x, y) &= R_{2N-2,2N}^{(4)}[\phi_{2N}^{(4)}(x)\phi_{2N-1}^{(4)}(y) - \phi_{2N}^{(4)}(y)\phi_{2N-1}^{(4)}(x)] \\
 &\quad + R_{2N-1,2N+1}^{(4)}[\phi_{2N-2}^{(4)}(x)\phi_{2N+1}^{(4)}(y) - \phi_{2N-2}^{(4)}(y)\phi_{2N+1}^{(4)}(x)] \\
 &\quad + (R_{2N-1,2N}^{(4)} - xP_{2N-1,2N}^{(4)})[\phi_{2N-2}^{(4)}(x)\phi_{2N}^{(4)}(y) - \phi_{2N-2}^{(4)}(y)\phi_{2N}^{(4)}(x)] \\
 &= \frac{1}{4\pi}[\cos(f_{2N}(\theta + \Delta\theta))\sin(f_{2N-2}(\theta)) - \cos(f_{2N}(\theta))\sin(f_{2N-2}(\theta + \Delta\theta))] \\
 &\quad + \frac{1}{4\pi}[\cos(f_{2N-2}(\theta + \Delta\theta))\sin(f_{2N}(\theta)) - \cos(f_{2N-2}(\theta))\sin(f_{2N}(\theta + \Delta\theta))] \\
 &\quad + \frac{\cos \theta}{4\pi \sin \theta}[\cos(f_{2N-2}(\theta + \Delta\theta))\cos(f_{2N}(\theta)) \\
 &\quad - \cos(f_{2N-2}(\theta))\cos(f_{2N}(\theta + \Delta\theta))]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{4\pi} [\sin[f_{2N-2}(\theta) - (f_{2N}(\theta + \Delta\theta))] + \sin[f_{2N}(\theta) - f_{2N-2}(\theta + \Delta\theta)] \\
 &\quad + \frac{\cos\theta \sin(2\theta)}{4\pi \sin\theta} [\sin[\Delta\theta f'_{2N}(\theta)]] \\
 &= -\frac{1}{4\pi} \left[\sin \left[\left(2N + \frac{a+b+1}{2} \right) \Delta\theta - 2\theta \right] + \sin \left[\left(2N + \frac{a+b+1}{2} \right) \Delta\theta + 2\theta \right] \right] \\
 &\quad + \frac{\cos\theta \sin(2\theta)}{4\pi \sin\theta} \left[\sin \left[\left(2N + \frac{a+b+1}{2} \right) \Delta\theta \right] \right] \\
 &\simeq -\frac{1}{2\pi} \sin[2N\Delta\theta][\cos(2\theta) - \cos^2\theta], \tag{6.28}
 \end{aligned}$$

where in the second step, we have dropped $O(N^{-1})$ terms. This gives us

$$\begin{aligned}
 S_{2N}^{(4)}(x, y) &= \frac{\sin(2N\Delta\theta)}{2\pi \Delta\theta \sin\theta}, \\
 &= \frac{\sin(2N(1-y^2)^{-1/2}\Delta y)}{2\pi \Delta y}, \quad |y| < 1. \tag{6.29}
 \end{aligned}$$

With $\Delta y \rightarrow 0$, we get the level density

$$S_{2N}^{(4)}(y, y) = \frac{N}{\pi \sqrt{1-y^2}}, \quad |y| < 1. \tag{6.30}$$

In the bulk of the spectrum, we get the ‘universal’ sine-kernel

$$\frac{S_{2N}^{(4)}(x, y)}{S_{2N}^{(4)}(y, y)} = \frac{\sin 2\pi r}{2\pi r}, \quad r = \Delta y S_{2N}^{(4)}(y, y). \tag{6.31}$$

6.3. Associated Laguerre SOP

Now we consider SOP (6.2) defined with respect to the associated Laguerre weight (2.7). As shown in [3, 4], $\pi_j^{(4)}(x) \equiv \pi_j(x)$ and $\pi_j'(x)$ can be written compactly in terms of associated Laguerre OP $L_j^{(a)}(x)$,

$$\pi'_{2m+1}(x) = L_{2m}^{(a)}(x), \quad m = 0, 1, \dots, \tag{6.32}$$

$$\pi'_{2m}(x) = L_{2m-1}^{(a)}(x) + \left(\frac{2m+a-1}{2m-1} \right) \pi'_{2m-2}(x), \quad m = 1, 2, \dots, \quad \pi'_0(x) = 0. \tag{6.33}$$

On integration, we find

$$\pi_{2m+1}(x) = -L_{2m+1}^{(a)}(x) + L_{2m}^{(a)}(x), \quad m = 0, 1, \dots, \tag{6.34}$$

$$\pi_{2m}(x) = -L_{2m}^{(a)}(x) + L_{2m-1}^{(a)}(x) + \left(\frac{2m+a-1}{2m-1} \right) \pi_{2m-2}, \quad m = 0, 1, \dots \tag{6.35}$$

For $a = 0$, (6.34) and (6.35) give back the results of [29], with the observation that any multiple of $\pi_{2m}(x)$ can be added to $\pi_{2m+1}(x)$. The normalization constant is given by

$$g_{2m}^{(4)} = g_{2m+1}^{(4)} = -h_{2m}^{(a)}. \tag{6.36}$$

The results (6.34) and (6.35) derive from (6.32) and (6.33) from the indefinite integral,

$$\int \mathbf{L}_j^{(a)}(x) dx = -L_{j+1}^{(a-1)}(x) = -L_{j+1}^{(a)}(x) + L_j^{(a)}(x),$$

the constants of integration in (6.34) and (6.35) being zero on skew-orthogonality with $\pi_1(x)$.

To obtain asymptotic formula for associated Laguerre SOP, we use (6.34) and (6.35) and the asymptotic formula for associated Laguerre OP (4.4) and (4.5). To avoid θ floating inside the argument, for a given x , we choose θ which effectively corresponds to $j = 2m + 1/2$ in (4.4). For ‘ a ’ arbitrary and real, ϵ a fixed positive number, we have for

$$x = (8m + 2a + 4) \cos^2 \theta, \quad \epsilon \leq \theta \leq \pi/2 - \epsilon m^{-\frac{1}{2}}, \quad \theta \equiv \theta_{2m+1/2}, \tag{6.37}$$

$$\sqrt{g_{2m}^{(4)} \phi_{2m}^{(4)}}(x) = -\frac{(2m)^{a/2}}{2} \left\{ \left[\frac{1}{2\sqrt{\pi m \cos \theta \sin^3 \theta}} \cos[f_{2m}(\theta)] + 1 + \frac{O(1)}{m\sqrt{x}} \right] \right\},$$

$$\sqrt{g_{2m}^{(4)} \phi_{2m+1}^{(4)}}(x) = \frac{(2m)^{a/2}}{\sqrt{m\pi \tan \theta}} \left[\sin[f_{2m}(\theta)] + \frac{O(1)}{\sqrt{mx}} \right], \tag{6.38}$$

$$\sqrt{g_{2m}^{(4)} \psi_{2m}^{(4)}}(x) = (2m)^{a/2} \left[\frac{1}{8\sqrt{\pi m \cos^3 \theta \sin \theta}} \sin[f_{2m}(\theta)] + \frac{O(1)}{m\sqrt{x}} \right], \tag{6.39}$$

$$\sqrt{g_{2m}^{(4)} \psi_{2m+1}^{(4)}}(x) = (2m)^{a/2} \left[\frac{1}{2} \sqrt{\frac{\tan \theta}{m\pi}} \cos[f_{2m}(\theta)] + \frac{O(1)}{m\sqrt{x}} \right], \tag{6.40}$$

with

$$\sqrt{g_{2m\pm 2}^{(4)} \phi_{2m\pm 2}^{(4)}}(x) = -\frac{(2m)^{a/2}}{2} \left\{ \left[\frac{1}{2\sqrt{\pi m \cos \theta \sin^3 \theta}} \cos[f_{2m}(\theta) \mp 4\theta] + 1 + \frac{O(1)}{m\sqrt{x}} \right] \right\}. \tag{6.41}$$

Similar relations hold for (6.38)–(6.40) as $m \rightarrow m \pm 1$. Here

$$f_{2m}(\theta) = (2m + 1 + a/2)(\sin 2\theta - 2\theta) + \frac{3\pi}{4}. \tag{6.42}$$

In deriving (6.37) we have used the large- m approximation

$$\left(\frac{8m - x}{2m} \right) \pi_{2m}(x) = -(L_{2m}^{(a)}(x) + L_{2m+1}^{(a)}(x)) - \frac{1}{2} \left(\frac{8m - x}{2m} \right) (2m)^{a/2} (w_a(x))^{-1/2}, \tag{6.43}$$

which follows from the three-term recursion and a sum rule for $L_j^{(a)}(x)$ [19].

6.4. Level-density and two-point correlation for associated Laguerre symplectic ensemble

In this subsection, we calculate the level density of associated Laguerre symplectic ensemble and show that in the bulk of the spectrum, the scaled or ‘unfolded’ [28] kernel function $(S_{2N}^{(4)}(x, y)/S_{2N}^{(4)}(y, y))$ is stationary and universal.

To study the kernel function (2.31), we need to calculate $R_{2N-1, 2N+1}^{(4)}$, $R_{2N-1, 2N}^{(4)}$, $R_{2N-2, 2N}^{(4)}$ and $P_{2N-1, 2N}^{(4)}$. For this, we use (2.18) and (2.19) for associated Laguerre weight,

$$x \frac{d}{dx} \phi_{2m+1}^{(4)}(x) = \sum_{j=2m-2}^{2m+2} P_{2m+1, j}^{(4)} \phi_j^{(4)}(x), \tag{6.44}$$

$$x^2 \frac{d}{dx} \phi_{2m+1}^{(4)}(x) = \sum_{j=2m-2}^{2m+3} R_{2m+1, j}^{(4)} \phi_j^{(4)}(x), \quad x^2 \frac{d}{dx} \phi_{2m}^{(4)}(x) = \sum_{j=2m-2}^{2m+2} R_{2m, j}^{(4)} \phi_j^{(4)}(x). \tag{6.45}$$

From which, using properties of OP, we get

$$P_{2m+1, 2m+2}^{(4)} = \sqrt{\frac{g_{2m+2}^{(4)}}{g_{2m}^{(4)}}} (m + 1), \quad R_{2m+1, 2m+3}^{(4)} = -\sqrt{\frac{g_{2m+2}^{(4)}}{g_{2m}^{(4)}}} (m + 1)(2m + 3), \tag{6.46}$$

$$R_{2m,2m+2}^{(4)} = -\sqrt{\frac{g_{2m+2}^{(4)}}{g_{2m}^{(4)}}(m+1)(2m+1)}, \quad R_{2m+1,2m+2}^{(4)} = \sqrt{\frac{g_{2m+2}^{(4)}}{g_{2m}^{(4)}}(m+1)(4m+a+4)}. \tag{6.47}$$

For $m = N - 1$, large N , we have $(g_{2N}^{(4)}/g_{2N-2}^{(4)}) \simeq 1$. Finally for large N , the recursion coefficients are given by

$$\begin{aligned} P_{2N-1,2N}^{(4)} &\sim N + O(1), & R_{2N-1,2N+1}^{(4)} &\sim -2N^2 + O(N), \\ R_{2N-2,2N}^{(4)} &\sim -2N^2 + O(N), & R_{2N-1,2N}^{(4)} &\sim 4N^2 + O(N). \end{aligned} \tag{6.48}$$

Also, since $g_{2N-2}^{(4)}/g_{2N}^{(4)} \simeq 1$, we can write

$$\begin{aligned} \phi_{2N}^{(4)}(x)\phi_{2N-1}^{(4)}(y) &= \frac{1}{\sqrt{g_{2N-2}^{(4)}g_{2N}^{(4)}}}\sqrt{w(x)w(y)}\pi_{2N}(x)\pi_{2N-1}(y) \\ &\simeq \frac{1}{g_{2N}^{(4)}}\sqrt{w(x)w(y)}\pi_{2N}(x)\pi_{2N-1}(y) \\ &= -\frac{1}{h_{2N}^a}\sqrt{w(x)w(y)}\pi_{2N}(x)\pi_{2N-1}(y) \\ &\simeq -(2N)^{-a}\sqrt{w(x)w(y)}\pi_{2N}(x)\pi_{2N-1}(y). \end{aligned} \tag{6.49}$$

Finally using the asymptotic results (6.37), (6.38), (6.41), defined in the θ interval $[\epsilon, \pi/2 - \epsilon N^{-\frac{1}{2}}]$, and the large m results (6.48) and (6.49) in the GCD formula (2.31), we get for

$$\begin{aligned} y &= (8N + 2a + 4) \cos^2 \theta, & x &= y + \Delta y, \\ y(y-x)S_{2N}^{(4)}(x, y) &= R_{2N-2,2N}^{(4)}[\phi_{2N}^{(4)}(x)\phi_{2N-1}^{(4)}(y) - \phi_{2N}^{(4)}(y)\phi_{2N-1}^{(4)}(x)] \\ &\quad + R_{2N-1,2N+1}^{(4)}[\phi_{2N-2}^{(4)}(x)\phi_{2N+1}^{(4)}(y) - \phi_{2N-2}^{(4)}(y)\phi_{2N+1}^{(4)}(x)] \\ &\quad + (R_{2N-1,2N}^{(4)} - xP_{2N-1,2N}^{(4)})[\phi_{2N-2}^{(4)}(x)\phi_{2N}^{(4)}(y) - \phi_{2N-2}^{(4)}(y)\phi_{2N}^{(4)}(x)] \\ &= -\frac{N}{2\pi \sin^2 \theta} \{ \sin[f_{2N-2}(\theta) - f_{2N}(\theta + \Delta\theta)] + \sin[f_{2N}(\theta) - f_{2N-2}(\theta + \Delta\theta)] \} \\ &\quad + \frac{N \cos 2\theta}{4\pi \cos \theta \sin^3 \theta} \{ \cos[f_{2N-2}(\theta + \Delta\theta)] \cos[f_{2N}(\theta)] \\ &\quad - \cos[f_{2N-2}(\theta)] \cos[f_{2N}(\theta + \Delta\theta)] \} \\ &= \left[\frac{N \cos 4\theta}{\pi \sin^2 \theta} - \frac{N \sin 4\theta \cos 2\theta}{4\pi \cos \theta \sin^3 \theta} \right] \sin \left(\frac{\partial f_{2N}}{\partial \theta} \Delta\theta \right) \\ &= 8N \cos^2 \theta \frac{[\cos 4\theta - \cos^2 2\theta]}{2\pi \sin^2 2\theta} \sin \left(\frac{\partial f_{2N}}{\partial \theta} \Delta\theta \right), \end{aligned} \tag{6.50}$$

which gives

$$\Delta y S_{2N}^{(4)}(x, y) = -\frac{\sin(8N \sin^2 \theta \Delta\theta)}{2\pi}.$$

Thus we get

$$S_{2N}^{(4)}(x, y) = \frac{\sin \left(\frac{1}{2}y^{-\frac{1}{2}}\sqrt{8N-y}\Delta y \right)}{2\pi \Delta y}, \quad 0 < y < 8N. \tag{6.51}$$

With $\Delta y \rightarrow 0$, we get the level density

$$S_{2N}^{(4)}(y, y) = \frac{1}{4\pi} \sqrt{\frac{8N - y}{y}}, \quad 0 < y < 8N. \quad (6.52)$$

Finally, in the bulk of the spectrum, we get the ‘universal’ sine-kernel (6.31).

7. Conclusion

In [3, 4], the authors prove ‘universality’ for the entire Jacobi ensembles of random matrices using SOP, which were written in terms of OP. The asymptotic properties of OP were used to obtain asymptotic formulae for SOP. Finally, the summation in $S_{2N}^{(\beta)}(x, y)$ was replaced by an integral for large N . However, using asymptotic results of the SOP to calculate the kernel function, before completing the sum lacks mathematical rigor.

In this paper, we have shown that SOP $\Phi_n^{(\beta)}(x)$ and $\Psi_n^{(\beta)}(x)$ corresponding to classical weight satisfy three-term recursion relations in the 2×2 quaternion space. Using this, we obtain the kernel functions $S_{2N}^{(\beta)}(x, y)$, $\beta = 1, 4$, for the entire family of finite-dimensional Jacobi ensembles of random $2N \times 2N$ matrices. As $N \rightarrow \infty$, we use the asymptotic results of the SOP in the range $[x_1 + \epsilon, x_2 - \epsilon]$, over which they are defined, to prove that in the bulk of the spectrum, the correlation functions are universal.

Here, we would like to mention that our GCD results are valid in the entire complex plane. Hence to study statistical properties of the eigenvalues of orthogonal and symplectic ensembles away from the bulk, we need to use Plancherel–Rotach-type formula for these polynomials defined outside the range $[x_1 + \epsilon, x_2 - \epsilon]$, something which has already been done for the unitary ensembles. We wish to come back to this in a later publication.

We would like to emphasize that the key step in deriving the asymptotic results of SOP is to obtain and solve finite-term recursion relations between SOP and OP. Recently, this technique has been used to obtain bulk asymptotics of SOP corresponding to quartic double well potential [7]. Till date, this seems the easier method to study the asymptotic behavior of SOP rather than solving the $2d \times 2d$ Riemann–Hilbert problem [6, 45].

Finally, a word on duality. Results in section 3 makes us wonder if the two families of SOP are really different or there exists a simple mapping between them. We have seen the existence of a simple relation (3.14) between SOP arising in the study of the associated Laguerre ensembles. The other ensembles do show a similar pattern, although we are unable to come out with a general formula. A deeper theoretical understanding is needed to obtain the mapping between these two families of SOP.

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